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REVOLUTION OF NEUTRAL BODY IN A CLOSED ELLIPSE UNDER GRAVITATIONAL FORCE OF ATTRACTION

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Abstract
A body of mass m revolves in an orbit through angle φ in time t, at a distance r from a much bigger mass M, with constant angular momentum $\boldsymbol{J}$, as:

$$\mathbf{J} = mr^2 \frac{d\varphi}{dt}\mathbf{k}$$

where $\boldsymbol{k}$ is a unit vector perpendicular to the plane of the orbit. The equation of orbit is:

$$\frac{1}{r} = A\cos(\varphi + \beta) + \frac{GMm^2}{J^2}$$

where G is the gravitational constant and amplitude A and phase angle β are determined from the initial conditions. The orbit of motion is a closed ellipse with the centre of force of attraction, the centre of gravity of the masses m and M, as the focus.

Keywords: Angular momentum, eccentricity, force, orbit, perihelion, revolution,

1 Introduction
Revolution of a body, round a centre of force of attraction, is the most common motion in the universe. This comes with revolution of planets round the Sun, binary stars round their centre of mass, a moon round a planet or an electron round the nucleus of an atom.

The German astronomer, Johannes Kepler [1, 2] early in the 17th century, formulated three laws, named after him, concerning the motions of planets. Kepler based his laws on astronomical data painstakingly collected in 30 years of observations by the Danish astronomer Tycho Brahe [3], to whom he was an assistant. Kepler's proposals broke with a centuries-old belief based on the Ptolemaic system advanced by the Alexandrian astronomer Ptolemy [4], in the 2nd century AD, and the Copernican system put forward by the Polish astronomer, Nicolaus Copernicus [5], in the 16th century.

The Ptolemaic cosmology postulated a geocentric universe in which the Earth was stationary and motionless at the centre of several concentric rotating spheres, which bore (in order of distance away from the Earth) the Moon, the planets and the stars. The major premises of the Copernican system are that the Earth rotates daily on its axis and revolves yearly round the Sun and that the planets also circle the Sun. Copernicus's heliocentric theories of planetary motion had the advantage of accounting for the daily and yearly motions of the Sun and stars and it neatly explained the observed motions of the planets. However, the reigning dogma in the 16th century, that of the Roman Catholic Church, was in favour of the Ptolemaic system and it abhorred the Copernican theory.

The Copernican theory had some modifications and various degrees of acceptance in the 16th and 17th centuries. The most famous Copernicans were the Italian physicist Galileo Galilei [6] and his contemporary, the astronomer Johannes Kepler [1, 2].

By December 1609 Galileo [7] had built a telescope of 20 times magnification, with which he discovered four of the moons circling Jupiter. This showed that at least some heavenly bodies move around a centre other than the Earth. By December 1910 Galileo [7] had observed the phases of Venus, which could be explained if Venus was sometimes nearer the Earth and sometimes farther away from the Earth, following a motion round the Sun.

In 1616, Copernican books were subjected to censorship by the Church [8]. Galileo was instructed to no longer hold or defend the opinion that the Earth moved. He failed to conform to the ruling of the Church and after the publication of his book titled *Dialogue on the Two Chief World Systems* [9], he was accused of heresy, compelled to recant his beliefs and then confined to house arrest. Galileo's *Dialogue* was ordered to be burned and his ideas banned.

The ideas contained in the *Dialogue* could not be suppressed by the Roman Catholic Church. Galileo's reputation continued to grow in Italy and abroad, especially after his final work. Galileo's final and greatest work is the book titled *Discourses Concerning Two New Sciences*, published in 1638. It reviews and refines his earlier studies of motion and, in general, the principles of mechanics. The book opened a road that was to lead Sir Isaac Newton [10] to the law of universal gravitation, which linked the planetary laws discovered by astronomer Kepler with Galileo's mathematical physics.

Kepler [1, 2] stamped the final seal of validity on the Copernican planetary system in three laws, viz.:

(i) *The paths of the planets are ellipses with the sun as one focus.*

(ii) *The line drawn from the sun to the planet sweeps over equal areas in equal time.*

(iii) *The square of the periods of revolution (T) of the different planets are proportional to the cube of their respective mean distances (r) from the sun* $(T^2 \propto r^3)$

The import of Kepler's first law is that there is no dissipation of energy in the revolution of a planet, in a closed orbit, round the Sun. Any change of kinetic energy is equal to the change of potential energy. The second law means that a planet revolves round the Sun with constant angular momentum, which is the case if there is no force perpendicular to the radius vector. From the third law $(T^2 \propto r^3)$ and the relationship between the centripetal force and speed $(F \propto v^2/r)$, it can be deduced that the force of attraction F on a planet is inversely proportional to the square of its distance from the Sun as discovered by Newton in 1687 [10]..

Kepler's laws played an important part in the work of the English astronomer, mathematician and physicist, Sir Isaac Newton [10]. The laws are significant for the understanding of the orbital paths of the moon, the natural satellite of the Earth, and the paths of the artificial satellites launched from space stations.

The purpose of this paper is to derive equation of the orbit of motion of bodies, such as the planets, like the Earth, revolving round a centre of force of attraction, the Sun. The starting points are Newton's universal law of gravitation and second law of motion.

2 Newton's universal law of gravitation.

The force of attraction $\mathbf{F}$ between two bodies of masses M and m, distance r apart in space, is given by Newton's universal law of gravitation:

$$\mathbf{F} = \frac{-GMm}{r^2}\hat{\mathbf{u}} \tag{1}$$

where G is the gravitational constant and $\hat{\boldsymbol{u}}$ is a unit vector in the radial direction. This force is extremely feeble that it is only noticeable in respect of huge masses like the moons, the planets and the Sun.

Two bodies under mutual attraction will revolve round their centre of mass with the larger mass M revolving in an inner orbit and the lighter mass m revolving in an outer orbit. The two bodies will revolve with the same angular velocity under equal and opposite forces of attraction. If M is very much larger than m as in the case of the Sun $(M = 2 \times 10^{30}\ kg)$ and the Earth $(m = 6 \times 10^{24}\ kg)$, the Sun may be considered as almost stationary at a point while the Earth revolves at a distance r from the centre of the Sun.

The gravitational force of attraction, on a moving body, is independent of its speed in a gravitational field. Newton's second law of motion, on a body of mass m moving at time t with speed v and acceleration dv/dt in the radial direction, gives:

$$\mathbf{F} = \frac{-GMm}{r^2}\hat{\mathbf{u}} = m\frac{dv}{dt}\hat{\mathbf{u}} \tag{2}$$

To obtain an expression for the acceleration (dv/dt)$\hat{\mathbf{u}}$, let us consider the revolution of a body round a centre of force of attraction.

3 Velocity and acceleration under a central motion

In Figure 1, the radius vector ***OP*** makes an angle φ with the ***OX*** axis in space. The position vector $\boldsymbol{r}$ of the point P**,** in the direction of unit vector $\hat{\boldsymbol{u}}$, (radial direction) and the velocity $\boldsymbol{v}$ at time t, are respectively given by the vector equations:

$$\mathbf{r} = r\hat{\mathbf{u}} \tag{3}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\hat{\mathbf{u}} + r\frac{d\hat{\mathbf{u}}}{dt} \tag{4}$$

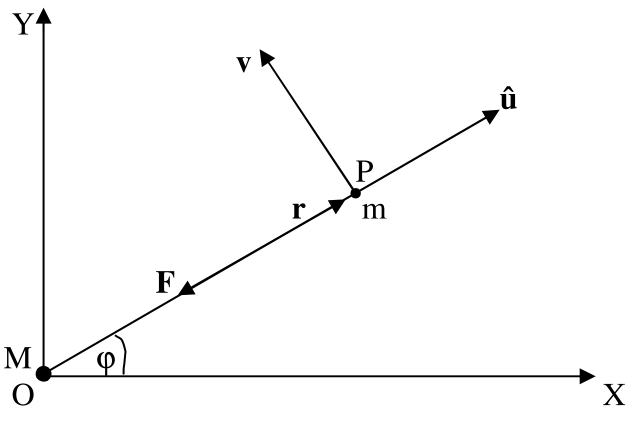

Figure 1. A body of mass m at a point P revolving anticlockwise, with angular displacement φ, in an orbit at velocity $\boldsymbol{v}$ under the attraction of a stationary mass M at a centre of attraction O.

For orbital motion in the X-Y plane of the Cartesian coordinates, $d\hat{\boldsymbol{u}}/dt$, the angular velocity, is given by the vector (cross) product:

$$\frac{d\hat{\mathbf{u}}}{dt} = \frac{d\varphi}{dt}\mathbf{k} \times \hat{\mathbf{u}} \tag{5}$$

The angle φ is the inclination of the radius vector ***OP*** from ***OX***, $\boldsymbol{k}$ is a constant unit vector, in the **Z**-direction, perpendicular to the orbital plane (out of the page in Figure 1) and $(d\varphi/dt)\boldsymbol{k}$ is the angular velocity. The velocity $\boldsymbol{v}$ is:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\hat{\mathbf{u}} + r\frac{d\varphi}{dt}\mathbf{k}\times\hat{\mathbf{u}} \tag{6}$$

The velocity in the radial $\hat{\boldsymbol{u}}$ direction is:

$$v_r\hat{\mathbf{u}} = \frac{dr}{dt}\hat{\mathbf{u}} \tag{7}$$

The acceleration (noting that $\boldsymbol{k}$ is a constant unit vector) is obtained, a vector in two orthogonal directions, as:

$$\frac{d\mathbf{v}}{dt} = \left\{\frac{d^2 r}{dt^2} - r\left(\frac{d\psi}{dt}\right)^2\right\}\hat{\mathbf{u}} + \left(2\frac{dr}{dt}\frac{d\psi}{dt} + r\frac{d^2\psi}{dt^2}\right)\mathbf{k}\times\hat{\mathbf{u}} \tag{8}$$

The acceleration, in the direction of force of attraction, is $a_r\hat{\boldsymbol{u}}$ in the radial direction, so that:

$$\frac{d\mathbf{v}}{dt} = \left\{\frac{d^2 r}{dt^2} - r\left(\frac{d\psi}{dt}\right)^2\right\}\hat{\mathbf{u}} = a_r\,\hat{\mathbf{u}} \tag{9}$$

4 Angular momentum

In equation (8), the force perpendicular to the radial direction is zero and, therefore, the acceleration of mass m is also zero in this $\boldsymbol{k}\times\hat{\boldsymbol{u}}$ direction, so that the equation gives:

$$m\left(2\frac{dr}{dt}\frac{d\varphi}{dt} + r\frac{d^2\varphi}{dt^2}\right)\mathbf{k}\times\hat{\mathbf{u}} = \frac{m}{r}\frac{d}{dt}\left(r^2\frac{d\varphi}{dt}\right)\mathbf{k}\times\hat{\mathbf{u}} = 0$$

This equation can be expressed in terms of angular momentum, a vector $\boldsymbol{J}$, as:

$$\frac{m}{r}\frac{d}{dt}\left(r^2\frac{d\varphi}{dt}\right)\mathbf{k}\times\hat{\mathbf{u}} = \frac{1}{r}\frac{d}{dt}\mathbf{J}\times\hat{\mathbf{u}} = 0$$

where

$$\mathbf{J} = mr^2\frac{d\varphi}{dt}\mathbf{k} \tag{10}$$

Here, $\boldsymbol{J}$ is the constant angular momentum at any point in the closed orbit, in accordance with Kepler's second law. Equation (10) and equation (9) will be used to derive the equation of the orbit of motion of the body at P in Figure 1.

5 Differential equation of motion

The centripetal acceleration on a body revolving through angle φ, under a central force, gives the accelerating force (equation 2 and 9) as:

$$\mathbf{F} = \frac{-GMm}{r^2}\hat{\mathbf{u}} = m\left\{\frac{d^2 r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2\right\}\hat{\mathbf{u}}$$

$$\frac{-GM}{r^2} = \frac{d^2 r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2 \tag{11}$$

This is a differential equation of motion in r and φ as functions of time t. We can reduce it to an equation of r as a function of φ only.

The angular momentum J of m, being a constant (Kepler's second law), gives:

$$J = mr^2 \frac{d\varphi}{dt} \tag{12}$$

Making the substitution $r = 1/u$ and with equation (12), gives:

$$\frac{dr}{dt} = \frac{dr}{du}\frac{d\varphi}{dt}\frac{du}{d\varphi} = \frac{-J}{m}\frac{du}{d\varphi} \tag{13}$$

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\frac{dr}{dt} = \frac{d\varphi}{dt}\frac{d}{d\varphi}\left(\frac{-J}{m}\frac{du}{d\varphi}\right) = \frac{-J^2u^2}{m^2}\frac{d^2u}{d\varphi^2} \tag{14}$$

Substituting equations (14) and (13) in equation (11) gives:

$$-\frac{J^2u^2}{m^2}\frac{d^2u}{d\varphi^2} - \frac{J^2u^3}{m^2} = -GMu^2$$

$$\frac{d^2u}{d\varphi^2} + u = \frac{GMm^2}{J^2} \tag{15}$$

Equation (15) is a second order differential equation with constant coefficients. Trying $u = A\exp(z\varphi)$ as a possible solution, we obtain the auxiliary equation:

$$z^2 + 1 = 0$$

.

$$z^2 = \sqrt{-1} = \pm j$$

The general solution of equation (15) is:

$$u = \frac{1}{r} = A\exp(j\varphi) + \frac{GMm^2}{J^2} \tag{16}$$

An appropriate solution is:

$$\frac{1}{r} = A\cos(\varphi + \beta) + \frac{GMm^2}{J^2} \tag{17}$$

where the amplitude A and phase angle β are determined from the initial conditions. If $\beta = 0$, equation (17) may be written as:

$$\frac{1}{r} = \frac{GMm^2}{J^2}\left(1 + \frac{AJ^2}{GMm^2}\cos\varphi\right) \tag{18}$$

This is an ellipse with the major axis along the X-axis. Equation (18) then becomes:

$$\frac{1}{r} = B\left(1 + \frac{A}{B}\cos\varphi\right) = B\left(1 + \eta\cos\varphi\right) \tag{19}$$

where $B = GMm^2/J^2$. Equation (19) gives an ellipse, in the polar coordinates, with eccentricity $\eta = A/B$. The ellipse is shown as *XYZW* in Figure 2.

In Figure 2, the lighter mass m revolves, in angle φ, at a point P, in a closed ellipse, at a distance r from a much heavier mass M at one focus F_1. The other focus F_2 is at a distance s from P. A property of an ellipse is that the distances $s + r = 2a$, the length of the major axis *ZX*. Other properties are obtained as the angle φ takes values 0, $\pi/2$ and π radians. The line *ZX* is the major axis, *WY* is the minor axis and the chords *CD* and *EG* are the latus rectums, with half length $l = a(1 - \eta^2) = 1/B$. The eccentricity of the ellipse is ratio of the distance between the foci (F_1F_2) to the length of the major axis *(ZX)*. For a circle, the two foci coincide and the eccentricity η is equal to zero.

In planetary motion, X, the point of closest approach to the Sun, is called the perihelion and Z, the point of farthest separation, is the aphelion. A planet moves faster as it approaches the Sun; it is fastest at the perihelion and slowest at the aphelion. At any time, a planet moves in such a way that the difference in kinetic energy between two points is equal to the difference in potential energy.

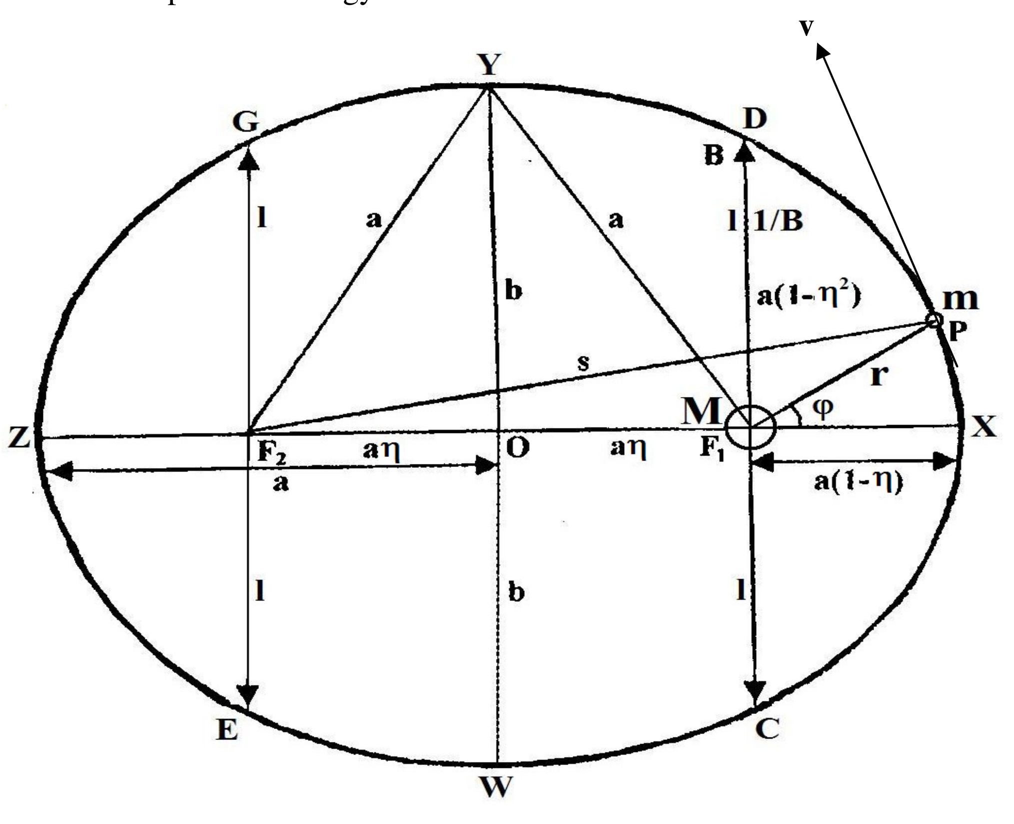

Figure 2. A body of lighter mass m revolving in an elliptic orbit, of eccentricity η, round a much heavier body of mass M at the centre of mass, the focus F_1,

6 Elliptic motion of an electrically charged body

It should be noted that in equation (15), the radiation component containing $du/d\varphi$, in the case of a charged body revolving under a central force, is missing. Where the masses M and m are electrically charged, there is a possibility of radiation. A charged body dissipates energy whenever it moves in the direction an electric field. Where there is radiation of energy, the ellipse is not closed but the perihelion (point X in Figure 2) rotates about an axis (point O in Figure 2). This rotation is called *precession of the perihelion*. There is no radiation where the revolution of a charged body is in a circle of constant radius.

7 Conclusion

The orbit of revolution of a neutral body, round a gravitational force of attraction, is a closed ellipse in one plane, without radiation of energy. Such a periodic motion may continue *ad infinitum*. In the event of the bodies carrying opposite charges there may be radiation of energy and precession of the perihelion, as observed in the revolution of Mercury, the planet nearest the Sun. The Sun may be positively charged and Mercury negatively charged and the high eccentricity of Mercury's orbit makes precession of its perihelion observable. If the Sun is positively charged, the planets (including the Earth) may as well be negatively charged but precessions of their perihelion are masked because of their larger distances from the Sun or because of their orbits being nearly circular.

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