# Remarks on the Equivalence of Inertial and Gravitational Masses and on the Accuracy of Einstein's Theory of Gravity 

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#### Abstract

This paper investigates the accuracy of Einstein's theory of gravity by studying the gravitational field near a spherically symmetric non-rotating massive body. The well-known Schwarzschild metric, which describes the space-time in the vicinity of such bodies, according to Einstein's theory of gravity, is compared with the new metric that is derived from first principles and without the use of Einstein's field equation. The basis for the derivation of the new metric is the new mass equivalence principle derived as a consequence of thought experiments and a slightly modified Newton's gravitational law written with the proper time and proper distance. The new theory of static gravity is therefore a scalar metric theory. The new metric predictions are evaluated and compared for accuracy with observations and with the predictions of the perihelion advance and the gravitational red shift of the Schwarzschild metric. It is found that an excellent agreement is obtained between the theory and observations and significant differences from the predictions of Schwarzschild metric are observed only in the vicinity of the Schwarzschild radius. The new metric has no problems related to the "black hole" geometry, has no coordinate pathologies, does not have the event horizon, and does not have the now famous singularity in the center of the black hole.


## INTRODUCTION

The recent advances in technology have enabled an unprecedented accuracy in testing and verification of Einstein's theory of gravity. The lunar Laser Ranging (LLR) experiments ${ }^{1}$ have provided new data for investigation of motion of the Earth-Moon system to a high degree of precision. The new data from the Pulsar, EXO 0748-676 ${ }^{2}$, have provided excellent gravitational red shift values that allowed precise determination of the Pulsar mass to radius ratio. Thus it seems that there is no need to doubt the accuracy of commonly accepted theory of gravity and seek the limits of its validity. However, it is difficult to judge how successful the theory is when there is no alternative by which to independently measure or evaluate the limits. It is necessary to patiently wait for observations and hope that some day large enough deviations will be uncovered that will not be possible to explain within the given framework. From the conceptual point of view of the existing theory of gravity, it makes little sense to derive the field equations using the condition that in the space where the filed is supposed to be the field's mass-energy is zero. This point has been thoroughly discussed in the literature, for example by Logunov ${ }^{3}$. Finally, there is still that persistent unpleasant singularity inside the center of the black hole that puts a black mark on the otherwise beautiful

[^0]and elegant theory and which needs to be removed by such contrived notions as the cosmic censorship hypothesis.

For these reasons this article will derive an alternate theory of static gravity that will be a simple scalar metric theory based on first principles and thought experiments. The Einstein's field theory of gravity will be purposely avoided in order not to contaminate the derivations and only its results, the Schwarzschild metric ${ }^{4}$, will be used for comparisons. Sometimes it pays to step back from the beaten path and investigate the problem from a new point of view and simplified to its essentials. This article will therefore focus only on two items, the planetary perihelion advance, and the gravitational red shift. The derived metric will thus be valid only for the static, spherically symmetric, non-rotating body, which is massive enough that the disturbance of orbiting planets can be neglected. The next section will address the equivalence of inertial and gravitational masses that is necessary for the development of the scalar theory of gravity. The mass equivalence will be derived from simple thought experiments.

## MOVING CLOCKS

This section illustrates four thought experiments from which a new conclusion about the equivalence of inertial and gravitational masses is drawn. The principal idea of the experiments is to construct two simple clocks and observe their ticking in two different frames of reference; first in a comoving frame of reference and then in the laboratory frame of reference. It is expected that the results of the observation will fully agree with the Special Theory of Relativity (STR), particularly with the Lorentz coordinate transformation.

The first clock is constructed such that only electrical forces drive the clock. Since it is well known how such forces transform from the laboratory to co-moving frame of reference, there should be no problem in obtaining an agreement with STR. The second clock, identical to the first, is constructed such that only the gravitational forces drive it. This will result in some choices that will have to be made in order to obtain an agreement with STR. The transformation of gravity from one frame of reference to another depends on the type of gravity theory that is used. However, no matter what the theory is, it is necessary to obtain an agreement with the Lorentz coordinate transformation, Lorentz Covariance (LC), since this is the fundamental and well-tested principle underlying the entire modern physics and any theory, in particular the theory of gravity, must be consistent with it.

In Fig.1, the drawing of a simple clock is shown. The clock consists of two rigid insulating plates, each carrying a uniformly distributed total charge Q and -Q respectively, which is attached to the plate's matrix. The plates are parallel to each other and are separated by a small distance "a". They are also parallel to the YZ plane. The plates are sufficiently large with their area equal to $A=W \cdot L$ and the distance between them is small so that the peripheral field non-uniformities can be safely neglected.


Fig.1. Experimental setup with charged parallel plates moving with velocity v in the Z direction and observed in the laboratory frame of reference.

With these assumptions, it is easy to calculate the time before the plates collide after they are released. From the Gauss law it holds that:

$$
\begin{equation*}
\oiint_{S} \varepsilon \vec{E} \cdot d \vec{S}=Q \tag{1}
\end{equation*}
$$

Since the enclosed Gaussian integrating surface $S$ can be suitably selected, closely following the plate surface A , it is easily seen that for the electrical field intensity E and from it for the force F , which one plate exerts on the other, it is:

$$
\begin{equation*}
F=\frac{Q^{2}}{\varepsilon \cdot 2 A} \tag{2}
\end{equation*}
$$

The same holds for the second charge acting on the first, so the total force with which the plates attract each other is twice the value given in Eq.2. Next, it is assumed that the plates have an inertial mass $m_{i}$ and that they are moving toward each other with a very low velocity, so that no relativistic effects apply. Ordinary Newton's second law can thus describe the motion and the time to plate's collision can thus be calculated to be:

$$
\begin{equation*}
t_{c}^{2}=\frac{\varepsilon \cdot m_{i} \cdot a \cdot A}{Q^{2}} \tag{3}
\end{equation*}
$$

The process of colliding can now be repeated at will and this device can be considered as a time-measuring clock. The inverse of the time to collision is the local clock rate.

In the next step, the above-described clock will be observed from the laboratory frame of reference and thus it will be assumed that both parallel plates are now moving in the Z direction with a constant velocity v . The laboratory observer at rest in the coordinate system XYZ will observe, in addition to charge, also currents that these two moving charged plates generate. The currents will cause an additional force to appear in the laboratory observer's coordinate system. From Ampere's law it holds that:

$$
\begin{equation*}
\oint_{P} \vec{H} \cdot d \vec{l}=v \frac{Q}{L} \tag{4}
\end{equation*}
$$

Again, since the integrating path P can be suitably selected, the H field is easily found to be:

$$
\begin{equation*}
H=v \cdot \frac{Q}{2 \cdot W \cdot L} \tag{5}
\end{equation*}
$$

The mutual force, with which the plates attract each other, according to the Lorentz force equation, is:

$$
\begin{equation*}
F=Q \cdot|\vec{E}+\vec{v} \times \vec{B}|=\frac{Q^{2}}{\varepsilon \cdot A}\left(1-\frac{v^{2}}{c^{2}}\right) \tag{6}
\end{equation*}
$$

where c is the speed of light. The time to collision of the moving plates as observed in the laboratory coordinate system is thus equal to:

$$
\begin{equation*}
t_{c}^{2}=\frac{\varepsilon \cdot \frac{m(r s t)_{i}}{\sqrt{1-v^{2} / c^{2}}} \cdot a \cdot A(r s t) \sqrt{1-v^{2} / c^{2}}}{Q^{2} \cdot\left(1-v^{2} / c^{2}\right)}=\frac{\varepsilon \cdot m(r s t)_{i} \cdot a \cdot A(r s t)}{Q^{2} \cdot\left(1-v^{2} / c^{2}\right)} \tag{7}
\end{equation*}
$$

This result is expected since it follows from STR after the Lorentz coordinate transformation. This phenomenon is the famous time dilation effect. It is nice to know that the simple clock works and produces the expected result. It is also important to note that the relativistic values for $\mathrm{m}_{\mathrm{i}}$ and A , as seen from the laboratory reference frame, have been substituted into Eq.7. The inertial mass $m_{i}$ has increased and the area A has shrunk in the Z direction but both with the same coefficient of proportionality, so the effects have cancelled each other and the numerator in Eq. 7 remained unchanged. The motion does not affect the distance " $a$ ", since it is perpendicular to the velocity vector $v$. Charge $Q$ also remains constant, since it is an absolute invariant.

Up to this point, everything functions well without any problems, so it is possible to proceed with the second clock experiments. The same clock is constructed but without charge Q and -Q , only the mutual gravitational force of the plates now facilitates the plate's attraction. It is assumed that the gravitational field intensity, and from this the force of attraction between the plates is:

$$
\begin{equation*}
F=\frac{4 \pi \cdot \kappa \cdot m_{g}^{2}}{A} \tag{8}
\end{equation*}
$$

where $\kappa$ is the gravitational constant. Using this force, the time to collision becomes:

$$
\begin{equation*}
t_{c}^{2}=\frac{m_{i} \cdot a \cdot A}{4 \pi \cdot \kappa \cdot m_{g}^{2}} \tag{9}
\end{equation*}
$$

This clock rate is the rate seen by the co-moving observer. The values of all the parameters in this formula are, of course, the rest reference frame values.

The next step, however, presents a problem. Unlike for the transformation of the electric filed from the moving to the laboratory coordinate system, there are no linear explicit formulas for the forces dependent on velocity available from the Einstein's filed equation of General Theory of Relativity (GTR). However, several authors have performed linearization for a week gravitational field and similar equations to Maxwell's equations have been presented in many publications. A relatively recent work on gravitomagnetic effects has been published by M.L. Ruggiero and A. Tartaglia ${ }^{5}$. By adapting their equations to the static case that is being studied here, the force of attraction for the moving plates can be expressed as:

$$
\begin{equation*}
\vec{F}=\frac{m_{g}}{m_{i}}\left(\vec{E}_{g}+\frac{1}{c} \vec{v} \times \vec{B}_{g}\right) \tag{10}
\end{equation*}
$$

where $\vec{E}_{g}$ and $\vec{B}_{g}$ are the gravito-static and the gravito-magnetic field intensities defined by the following differential equations:

$$
\begin{gather*}
\vec{\nabla} \cdot \vec{E}_{g}=-4 \pi \cdot \kappa \cdot \rho_{g}  \tag{11}\\
\vec{\nabla} \times \frac{1}{2} \vec{B}_{g}=-\frac{1}{c} \cdot 4 \pi \cdot \vec{j}_{g} \tag{12}
\end{gather*}
$$

with $\rho_{g}$ and $\vec{j}_{g}$ representing the mass density and the mass current density respectively. By using the Gauss and Stokes formulas, as in the previous case of the electric and magnetic fields, these equations can be easily solved for the mass configuration that is being investigated here and the formula for the time to collision derived. This becomes:

$$
\begin{equation*}
t_{c}^{2}=\frac{\frac{m(r s t)_{i}}{\sqrt{1-v^{2} / c^{2}}} \cdot a \cdot A(r s t) \sqrt{1-v^{2} / c^{2}}}{4 \pi \cdot \kappa \cdot m(r s t)_{g}^{2} \cdot \frac{\left(1-2 \cdot v^{2} / c^{2}\right)}{\left(1-v^{2} / c^{2}\right)}} \tag{13}
\end{equation*}
$$

It should be noted that Eq. 10 was derived only for small velocities and week gravitational field. These conditions are satisfied in this experiment. The velocity in the direction of plates' attraction and the plates' mass can be assumed arbitrarily small. No restriction is necessary for the velocity in the Z direction. From the result derived in Eq.13, it can be concluded that GTR for the week gravitational fields and small velocities is approximately consistent with the Lorentz coordinate transformation, but Eq. 13 is not a good fit for larger velocities. The well-know factor of two from the left hand side of Eq. 12 now appears in one of the velocity factor brackets of Eq. 13 and spoils the LC. This is a seldom-discussed problem among several pointed out by Logunov ${ }^{3}$. For this reason a different approach to the theory of static gravitational filed will be followed in this article. In the following derivations, it will be assumed that the inertial and the gravitational masses depend on velocity differently according to the following equations:

$$
\begin{align*}
& m_{g}=m(r s t)_{g} \sqrt{1-\frac{v^{2}}{c^{2}}}  \tag{14}\\
& m_{i}=m(r s t)_{i} \frac{1}{\sqrt{1-\frac{v^{2}}{c^{2}}}} \tag{15}
\end{align*}
$$

and no velocity dependent gravito-magnetic forces are present. The Einstein's mass equivalence between the inertial and the gravitational masses is thus maintained only at rest. By substituting the above equations into Eq. 9 the result becomes:

$$
\begin{equation*}
t_{c}^{2}=\frac{\frac{m(r s t)_{i}}{\sqrt{1-v^{2} / c^{2}}} \cdot a \cdot A(r s t) \sqrt{1-v^{2} / c^{2}}}{4 \pi \cdot \kappa \cdot m(r s t)_{g}^{2}\left(1-v^{2} / c^{2}\right)}=\frac{m(r s t)_{i} \cdot a \cdot A(r s t)}{4 \pi \cdot \kappa \cdot m(r s t)_{g}^{2}\left(1-v^{2} / c^{2}\right)} . \tag{16}
\end{equation*}
$$

This result now fully follows STR and is LC for all velocities. As a result it will be possible to use the new equivalence principle to build a covariant scalar metric theory of static gravitational filed based on Newton's gravitational law. In the following sections, it will be also assumed that $\mathrm{m}(\mathrm{rst})_{\mathrm{g}}=\mathrm{m}(\mathrm{rst})_{\mathrm{i}}=\mathrm{m}$. The different dependence of gravitational mass on velocity may seem strange and counterintuitive at first, however, if this is an universal dependency, all falling bodies will obey it and no violation of the Galileo free fall experiment will be observed. It is also clear that the free falling bodies will follow the geodesic curves regardless of their gravitational mass change. To the best of the author's knowledge no mass equivalence tests have been conducted or designed yet to test the mass equivalence for bodies that are moving at very high relative velocities. There seem to be no experimental support for the Einstein's mass equivalence other than for bodies at rest or at small relative velocities.

There are many consequences of this gravitational mass dependency on velocity, such as no gravitational interaction of photons moving in the same direction, as predicted by one of the quantum theories of gravity ${ }^{6}$. All particles moving with the velocities close to the sped of light will have no mutual gravitational interaction including neutrinos and gravitons. The difference between the inertial and the gravitational masses has also been predicted elsewhere ${ }^{7}$, based on the thermodynamic considerations. However, the discussion of these interesting topics is beyond the scope of this article and cannot be addressed here any further. Since this articles focuses only on the perihelion advance and the gravitational red shift, as mentioned in the introduction, the confirmation of the new mass equivalence principle will thus be left to the agreement of the derived results with observations.

To summarize the main conclusion of this section: it can be stated that as viewed from the laboratory coordinate system, the gravitational mass of the moving body, which moves with a constant velocity v , changes with the velocity such that the product $\mathrm{m}_{\mathrm{g}} \mathrm{m}_{\mathrm{i}}=\mathrm{m}^{2}$ remains constant. The topic of the next section will be the confirmation of the new mass equivalence principle by calculating the Mercury's perihelion advance.

## MERCURY'S PERIHELION ADVANCE

It is well known that the perihelion advance is a curved space-time phenomenon and the domain of GTR. The metric for the curved space-time, however, has not been introduced yet. The thought experiments were performed only within the bounds of STR. Nevertheless, it is interesting to see what effect the new mass equivalence principle has on this phenomenon.

Considering the Sun with a large mass $\mathrm{M}_{\mathrm{s}}$ being placed in the origin of the XYZ coordinate system, neglecting the mass of Mercury relative to the mass of Sun, and choosing the orbital plane the XY plane, it is possible to write the following two component equations that follow directly from Newton's second and gravitational laws:

$$
\begin{align*}
& \frac{d}{d t}\left[m_{i} \cdot \frac{d}{d t}(r \cdot \cos \varphi)\right]=-\kappa \cdot \frac{m_{g} \cdot M_{s}}{r^{2}} \cdot \cos \varphi  \tag{17}\\
& \frac{d}{d t}\left[m_{i} \cdot \frac{d}{d t}(r \cdot \sin \varphi)\right]=-\kappa \cdot \frac{m_{g} \cdot M_{s}}{r^{2}} \cdot \sin \varphi \tag{18}
\end{align*}
$$

By multiplying Eq. 17 with $\sin \varphi$ and Eq. 18 with $\cos \varphi$, subtracting the results, and after some simple algebraic manipulations, it is possible to derive the first integral of motion, corresponding to the conservation of angular momentum, in the form:

$$
\begin{equation*}
m_{i} \cdot r^{2} \cdot \frac{d \varphi}{d t}=\alpha \tag{19}
\end{equation*}
$$

where it was, of course, considered that $\mathrm{m}_{\mathrm{i}}$ and $\mathrm{m}_{\mathrm{g}}$ both depend on velocity. The introduced constant $\alpha$ is the constant of integration representing the angular momentum. Similarly, after some algebra, it is possible to derive the following equation for r :

$$
\begin{equation*}
\frac{1}{m_{i}} \cdot \frac{d m_{i}}{d t} \cdot \frac{d r}{d t}+\frac{d^{2} r}{d t^{2}}-r \cdot\left(\frac{d \varphi}{d t}\right)^{2}=-\kappa \cdot \frac{m_{g} \cdot M_{s}}{m_{i} \cdot r^{2}} \tag{20}
\end{equation*}
$$

This equation can be simplified, as is typically done ${ }^{4}$, by substituting $u$ for $1 / r$ and by using Eq.19. The result becomes:

$$
\begin{equation*}
\frac{d^{2} u}{d \varphi^{2}}+u=\kappa \cdot \frac{m_{g} \cdot m_{i} \cdot M_{s}}{\alpha^{2}} \tag{21}
\end{equation*}
$$

The solution of this equation leads to a classical result, describing elliptical orbits of Newtonian mechanics, when all the masses are considered constant.

There have been attempts in the past ${ }^{8}$ to calculate the Mercury's perihelion advance based strictly on STR assuming identical relativistic mass corrections for both the inertial and gravitational masses. However, the calculations yielded only a fraction of the observed value. This clearly indicates a problem but further attempts to resolve this discrepancy were unfortunately abandoned in favor of the GTR solution. STR should provide either a correct result or a zero result. A partial agreement is not reasonable or
acceptable. However, when the new assumption about the velocity dependence of the gravitational mass is used, the perihelion advance is zero. The right hand side of Eq. 21 becomes a constant again as in the Newtonian case. This is a very encouraging result clearly indicating that the assumption about the gravitational mass dependency on velocity is correct. Since no curved space-time was yet assumed the advance of the perihelion must be correctly calculated as zero. This result now becomes consistent with STR. The curved space-time concept is addressed in the next section.

## CURVED SPACE-TIME

To proceed further in investigation of the structure of the new space-time, where the gravitating mass depends on velocity according to Eq. 14 , it will be necessary to use a more powerful analytical tool. Let's consider again a central gravitating body placed in the origin of the XYZ coordinate system and find the Lagrangian for the motion of a small test body. Newton's second law and Newton's gravitation law lead to the following equation:

$$
\begin{equation*}
\frac{d}{d t}\left(m_{i} \frac{d x}{d t}\right)=-m_{g} \cdot \frac{\kappa \cdot M_{s}}{x^{2}} \tag{22}
\end{equation*}
$$

where, for simplicity, it was considered that the test body moves only along the X direction. The generalization to spherical coordinates and any arbitrary motion is a simple matter, but unnecessarily clutters the notation. By using the expressions from Eq. 14 and Eq. 15 for the $m_{g}$ and $m_{i}$ and by introducing the proper time $d \tau=d t \cdot \sqrt{1-v^{2} / c^{2}}$, it is possible to write the following two equations instead of Eq. 22 .

$$
\begin{gather*}
\frac{d}{d \tau}\left(m \cdot \frac{d x}{d \tau}\right)=-m \cdot \frac{\kappa \cdot M_{s}}{x^{2}}  \tag{23}\\
\left(\frac{c d t}{d \tau}\right)^{2}-\left(\frac{d x}{d \tau}\right)^{2}=c^{2} \tag{24}
\end{gather*}
$$

The rest mass m can be factored out from Eq. 23 and this signifies that the motion is independent of mass. Every massive body will follow the same trajectory regardless of its mass. The new mass equivalence principle made this possible and transformed the Newton's gravitational law into a form that is LC compatible. However, the domain is still a flat space-time as indicated by Eq. 24 and in such a flat spacetime it is not possible to construct the desired Lagrangian. To proceed further it is necessary to transfer the considerations to a curved space-time. It is interesting to note that the LC of Newton's gravitational law clearly demands a curved space-time.

The first step in generalization is to replace the flat space proper time by the curved space proper time. As a second step it will be assumed that the hypothesis of locality ${ }^{9}$ is valid. This means that the gravity can be locally transformed out by a free fall of the coordinate system with the body and that at any instant the coordinate transformation from the moving to the laboratory coordinate system is LC. Finally, it
will be considered that since the central gravitating mass $M_{s}$ is stationary and spherically symmetric, the metric of the curved space-time will also be spherically symmetric and will be time independent. To find the laboratory coordinate system Lagrangian for the motion of a test body in this static gravitational field, the Lagrangian can be considered in the following general form as described in appendix A :

$$
\begin{equation*}
L=e^{A(x)}\left(\frac{c d t}{d \tau}\right)^{2}-e^{B(x)}\left(\frac{d x}{d \tau}\right)^{2} \tag{25}
\end{equation*}
$$

where the functions $\mathrm{A}(\mathrm{x})$ and $\mathrm{B}(\mathrm{x})$ are any arbitrary functions of coordinate x satisfying certain conditions as also described in appendix A. A beautiful theorem that proves this assertion quite generally for all spacetime coordinates can be found in the literature ${ }^{10}$. The exponential factors in Eq. 25 represent the metric coefficients. The Lagrangian in Eq. 25 has now enough free coefficients to accommodate Newton's gravitational law. To find the equations of motion and the space-time metric, it is necessary to solve the following set of equations:

$$
\begin{gather*}
c^{2}=e^{A(x)}\left(\frac{c d t}{d \tau}\right)^{2}-e^{B(x)}\left(\frac{d x}{d \tau}\right)^{2}  \tag{26}\\
\frac{d^{2} x}{d \tau^{2}}=-\frac{\kappa \cdot M_{s}}{x^{2}} \tag{27}
\end{gather*}
$$

and the Euler-Lagrange (EL) equations for both the time and the space coordinates that follow from the Lagrangian:

$$
\begin{equation*}
\frac{d}{d \tau}\left(\frac{\partial L}{\partial \frac{d t}{d \tau}}\right)=\frac{\partial L}{\partial t}, \quad \quad \frac{d}{d \tau}\left(\frac{\partial L}{\partial \frac{d x}{d \tau}}\right)=\frac{\partial L}{\partial x} \tag{28}
\end{equation*}
$$

Eq. 26 is the relativistic coordinate constraint and Eq. 27 is Eq. 23 where the rest mass was factored out. More complex force dependence on x is also possible to consider here. For example, a static cosmology with the known radial mass distribution would lead to a different function for the second derivative.

To find the solution of this system of equations is not difficult, but somewhat tedious. Certain precautions in the procedure of finding the solution must also be observed. It proves beneficial to start with EL equation related to the time coordinate, followed by the coordinate constraint equation, and finally followed by solving the last EL equation. This procedure thus first leads to:

$$
\begin{equation*}
\frac{d}{d \tau}\left(2 e^{A(x)} c^{2} \frac{d t}{d \tau}\right)=0 \tag{29}
\end{equation*}
$$

Eq. 29 can be easily integrated and using the condition that the space-time at infinity is flat, it is possible to write:

$$
\begin{equation*}
\frac{d t}{d \tau}=e^{-A(x)} \tag{30}
\end{equation*}
$$

This result can then be used with Eq. 26 to obtain:

$$
\begin{equation*}
\left(\frac{d x}{d \tau}\right)^{2}=e^{-B(x)} c^{2}\left(e^{-A(x)}-1\right) \tag{31}
\end{equation*}
$$

This procedure has now yielded all the first derivatives of coordinates and ensured that the relativistic constraint and thus LC is satisfied. Finally, the second EL equation leads to:

$$
\begin{equation*}
-\frac{d}{d \tau}\left(2 e^{B(x)} \frac{d x}{d \tau}\right)=\left[A^{\prime}(x)-B^{\prime}(x)\right] \cdot e^{-A(x)} c^{2}+B^{\prime}(x) \cdot c^{2} \tag{32}
\end{equation*}
$$

where the prime represents $\mathrm{d} / \mathrm{dx}$. By carrying out the differentiation and using Eq. 31 and Eq. 27 , it is possible, after some algebra, to derive the following simple result:

$$
\begin{equation*}
c^{2} B^{\prime}(x)+2 \cdot e^{B(x)} \frac{\kappa \cdot M_{s}}{x^{2}}=\left[A^{\prime}(x)+B^{\prime}(x)\right] \cdot e^{-A(x)} c^{2} \tag{33}
\end{equation*}
$$

The use of Eq. 27 for the second derivative of coordinate x in Eq. 32 is the crucial step in obtaining the correct solution. If the second derivative were, for example, calculated from Eq. 31 the result would be that any functions $\mathrm{A}(\mathrm{x})$ and $\mathrm{B}(\mathrm{x})$ satisfy the equations. This is understandable, since all the free falling coordinate systems, which transform out the force of gravity, are equivalent. The system of equations for the functions $A(x)$ and $B(x)$ without Eq. 27 is thus not well defined. To specify the second derivative is, therefore, an interesting necessary and a crucial requirement for finding the functions $\mathrm{A}(\mathrm{x})$ and $\mathrm{B}(\mathrm{x})$. Appendix A explains that for a correct relativistic Lagrangian in the laboratory coordinate system it can be also specified that:

$$
\begin{equation*}
A(x)=-B(x) \tag{34}
\end{equation*}
$$

This condition actually guarantees that the Jacobian of the coordinate transformation from the free falling test body coordinate system to the laboratory coordinate system is equal to unity. Using this condition in Eq. 33 leads to a simple differential equation for $B(x)$. This is easily solved when the initial condition of flat space at large distances is again used:

$$
\begin{equation*}
e^{B(x)}=\frac{1}{1-\frac{2 \cdot \kappa \cdot M_{s}}{c^{2} \cdot x}} \tag{35}
\end{equation*}
$$

The curved space-time metric and the relativistic Lagrangian for a test body moving in the static gravitational field of the central gravitating body, when converted to spherical coordinates, are thus as follows:

$$
\begin{gather*}
(d s)^{2}=\left(1-\frac{2 \cdot \kappa \cdot M_{s}}{c^{2} \cdot r}\right) \cdot(c d t)^{2}-\left(1-\frac{2 \cdot \kappa \cdot M_{s}}{c^{2} \cdot r}\right)^{-1}(d r)^{2}-r^{2}\left(d \vartheta^{2}+\sin ^{2} \vartheta \cdot d \varphi^{2}\right),  \tag{36}\\
L=\left(1-\frac{2 \cdot \kappa \cdot M_{s}}{c^{2} \cdot r}\right) \cdot\left(\frac{c d t}{d \tau}\right)^{2}-\left(1-\frac{2 \cdot \kappa \cdot M_{s}}{c^{2} \cdot r}\right)^{-1}\left(\frac{d r}{d \tau}\right)^{2}-r^{2}\left(\left(\frac{d \vartheta}{d \tau}\right)^{2}+\sin ^{2} \vartheta \cdot\left(\frac{d \varphi}{d \tau}\right)^{2}\right) \tag{37}
\end{gather*}
$$

This is exactly the Schwarzschild solution of Einstein's theory of gravitation for an empty space. It is important to note that it was not necessary to use assumptions from which the Einstein's field equation is
derived. The key factor in finding the correct space-time metric was Newton's gravitation law written with the proper time as given in Eq. 27 . This was enabled by the use of the new mass equivalence principle. It is also apparent that for a different gravitational law a different metric would be obtained. These facts thus establish a strong link between Newton's gravitation law and the Schwarzschild metric of the curved spacetime. This result thus justifies the choice made in Eq. 14 about the gravitational mass dependency on velocity, instead of introduction of gravito-magnetic fields or other theories of gravity, and proves its correctness. The Einstein's theory of gravity is therefore consistent with Newton's gravitation law if the new mass equivalence principle is used.

The presented derivation of the Schwarzschild solution from the new mass equivalence principle should thus be a successful conclusion to the new theory and a crowning moment. However, there is a subtle flaw in the derivation even when a presumably correct result was obtained. When the transfer to the curved coordinate system was made, by using Lagrangian for the curved space-time, Newton's gravitation law was left unchanged. It is extremely unlikely that Newton's gravitational law would hold in coordinate distances. The coordinate distances in curved space-time, unlike in the flat space-time, are not physical quantities but only scaffolding for mapping the space. This casts a great suspicion on the validity of the Schwarzschild solution and the whole concept of Einstein's theory of gravity.

## NEW SPACE-TIME METRIC

In order to be consistent when working within the curved space-time domain, it is necessary to use Newton's law written in the curved space-time coordinates. It is necessary to use the proper time and the proper distance in the equation. It is strange that in Eq.27, which clearly leads to Schwarzschild solution of GTR, only the time was replaced by the proper time, but the coordinate distance was left alone. This peculiarity will be corrected in this section. The relationship between the coordinate and the proper distance is easily found and together with Newton's equation substituted into Eq.33. The goal is to obtain the new and correct metric for the space-time of the central gravitating mass.

It will be assumed that the proper distance $\rho(\mathrm{x})$ is a function of only the coordinate distance x for the static case that is considered here. Newton's equation written in terms of the proper distance and the proper time is as follows:

$$
\begin{equation*}
\frac{d^{2} \rho(x)}{d \tau^{2}}=-\frac{\kappa \cdot M_{s}}{\rho^{2}(x)} \tag{38}
\end{equation*}
$$

From Eq. 26 , the proper distance is defined by the following differential equation:

$$
\begin{equation*}
d \rho(x)=e^{\frac{B(x)}{2}} d x \tag{39}
\end{equation*}
$$

By carrying through the differentiation as indicated in Eq. 38, the following relation is obtained:

$$
\begin{equation*}
\frac{d^{2} \rho(x)}{d x^{2}} \cdot\left(\frac{d x}{d \tau}\right)^{2}+\frac{d \rho(x)}{d x} \cdot \frac{d^{2} x}{d \tau^{2}}=-\frac{\kappa \cdot M_{s}}{\rho^{2}(x)} \tag{40}
\end{equation*}
$$

In the next step, differentiating Eq. 39 with respect to x , substituting the result into Eq. 40 , and using Eq. 31 , the result, after some rearrangement, becomes:

$$
\begin{equation*}
\frac{d B}{d x} \cdot c^{2}\left(e^{B(x)}-1\right)+2 \cdot e^{B(x)} \cdot \frac{d^{2} x}{d \tau^{2}}=-\frac{2 \cdot \kappa \cdot M_{s}}{\rho^{2}(x)} \cdot e^{\frac{B(x)}{2}} \tag{41}
\end{equation*}
$$

From Eq. 31 also follows that:

$$
\begin{equation*}
2 \cdot e^{B(x)} \frac{d^{2} x}{d \tau^{2}}=c^{2} \frac{d B(x)}{d x} \tag{42}
\end{equation*}
$$

Eliminating the second derivative of $x$ from these two equations results in the following:

$$
\begin{equation*}
\frac{d B(\rho)}{d \rho} \cdot e^{B(\rho)}=-\frac{2 \cdot \kappa \cdot M_{s}}{c^{2} \rho^{2}} \tag{43}
\end{equation*}
$$

This equation can now be easily integrated. Assuming again that at large distances from the origin the space-time is flat, the result for $B$, expressed as a function of the proper distance, using the Schwarzschild radius to simplify the notation, becomes:

$$
\begin{equation*}
e^{B(\rho)}=1+\frac{R_{s}}{\rho} \tag{44}
\end{equation*}
$$

where:

$$
\begin{equation*}
R_{s}=\frac{2 \cdot \kappa \cdot M_{s}}{c^{2}} \tag{45}
\end{equation*}
$$

From this result the new metric line element for the spherically symmetric and static gravitational filed should be:

$$
\begin{equation*}
(d s)^{2}=\left(1+\frac{R_{s}}{\rho(r)}\right)^{-1}(c d t)^{2}-\left(1+\frac{R_{s}}{\rho(r)}\right) \cdot(d r)^{2}-R^{2}(\rho) \cdot\left(d \vartheta^{2}+\sin ^{2} \vartheta \cdot d \varphi^{2}\right) \tag{46}
\end{equation*}
$$

where a Radius function $R(\rho)$ has been introduced into the metric in accordance with the theorem derived in reference ${ }^{10}$. Unfortunately, Newton's law will not help to determine the Radius function in the metric line element and some other physical reasoning will have to be used to find it. It will be considered that the Radius function, or more precisely the metric coefficient standing by the angular coordinates, can be determined by minimizing the energy of the gravitational field in the space around the gravitating body. This assumption is in a sharp contrast with the Einstein's approach, where the Riemannian curvature instead of the field energy is minimized.

It will be assumed that the metric coefficient standing by the coordinate distance $r$ has generally the following form:

$$
\begin{equation*}
e^{B(r)}=1+R_{s} f(r) \tag{47}
\end{equation*}
$$

The static gravitational field energy stored in the space around the central gravitating body can be expressed, up to a multiplicative constant, as follows:

$$
\begin{equation*}
W \approx \int_{0}^{\infty} \frac{f^{\prime 2} \cdot R^{2}(r) \cdot d r}{1+R_{s} f(r)} \tag{48}
\end{equation*}
$$

where the prime signifies the derivative $\mathrm{d} / \mathrm{dr}$. The metric coefficient introduced in the denominator of the fraction behind the integration sign is necessary, since the gravitational filed intensity transforms as a covariant vector. The Radius function will be found by finding the extremum of W . This is accomplished by setting the variation of W to zero, $\delta \mathrm{W}=0$. The corresponding EL equation for this variational problem is:

$$
\begin{equation*}
\frac{d}{d r}\left(\frac{2 \cdot f^{\prime} \cdot R^{2}}{1+R_{s} f}\right)=-\frac{f^{\prime 2} R^{2} \cdot R_{s}}{\left(1+R_{s} f\right)^{2}} \tag{49}
\end{equation*}
$$

Solution of Eq. 49 is easily found to be:

$$
\begin{equation*}
R^{2}(r)=-\frac{\sqrt{1+R_{s} f(r)}}{f^{\prime}(r)} \tag{50}
\end{equation*}
$$

By expressing this result as function of the proper distance $\rho$, Eq. 50 simplifies to read:

$$
\begin{equation*}
R^{2}(\rho)=\frac{-1}{\dot{f}(\rho)} \tag{51}
\end{equation*}
$$

where the dot represents the derivative $\mathrm{d} / \mathrm{d} \rho$. By inspection, it is clear that the Radius function derived from the metric line element given in Eq. 46 is simply $R(\rho)=\rho$. This is encouraging, since this is what would normally be expected. Unfortunately, when this metric is used to evaluate the Mercury's perihelion advance, the result is zero. From this unexpected result it was concluded that Newton's gravitational law have to be modified. The formula in Eq. 44 suggests that this may represent only the first two terms of a power series expansion of some more complex relation. One such relation could, for example, be a geometric series as follows:

$$
\begin{equation*}
e^{B(\rho)}=\frac{1}{1-\frac{R_{s}}{\rho}}=1+\frac{R_{s}}{\rho}+\left(\frac{R_{s}}{\rho}\right)^{2}+\left(\frac{R_{s}}{\rho}\right)^{3}+\ldots \tag{52}
\end{equation*}
$$

This would bring us back to the existence of black holes and event horizons, the very thing this article is trying to show that do not exist. Nevertheless, it is interesting to calculate the Radius function for this metric. It is simply $R(\rho)=\left(\rho-R_{s}\right)$. This is an interesting result, actually very intuitive, which suggests that the classical black hole region needs to be excluded from the space when the gravitational space energy is calculated. Unfortunately when this metric is used for the calculation of Mercury's perihelion advance, the result is twice the observed value. It is interesting to note that the classical result is obtained only when the Radius function, which does not correspond to the minimum of the static gravitational field energy, is
used together with coordinate instead of proper distances. This is clearly again a nonphysical result for the Einstein's theory of gravity.

Finally, as will be shown in the next section, the correct result for the Mercury's perihelion advance is obtained when the metric coefficient has additional higher order terms in the metric element expansion starting with $1 / 2\left(R_{s} / \rho\right)^{2}$. Newton's gravitational law needs therefore a modification to agree with observations. To add more expansion terms to the metric coefficient is possible and finally arrive at a simple analytic formula for $B(\rho), B(\rho)=R_{s} / \rho$. Such a metric predicts the same perihelion advance as the metric with only the second order term within the accuracy of current observations. Therefore, it will be assumed that the extrapolated expression correctly describes the space-time of the studied problem.

The metric line element and the corresponding Lagrangian for the space-time in the vicinity of a gravitating non-rotating body, which agree with observations, are therefore as follows:

$$
\begin{gather*}
d s^{2}=e^{\frac{-R_{s}}{\rho}}(c d t)^{2}-e^{\frac{R_{s}}{\rho}} d r^{2}-\rho^{2} e^{\frac{-R_{s}}{\rho}}\left(d \vartheta^{2}+\sin ^{2} \vartheta \cdot d \varphi^{2}\right)  \tag{53}\\
L=e^{\frac{-R_{s}}{\rho}}\left(\frac{c d t}{d \tau}\right)^{2}-e^{\frac{R_{s}}{\rho}}\left(\frac{d r}{d \tau}\right)^{2}-\rho^{2} e^{\frac{-R_{s}}{\rho}}\left(\left(\frac{d \vartheta}{d \tau}\right)^{2}+\sin ^{2} \vartheta \cdot\left(\frac{d \varphi}{d \tau}\right)^{2}\right) \tag{54}
\end{gather*}
$$

It is clear that this metric does not have any pathology at the Schwarzschild radius, does not have any censored singularities with event horizons and that it covers the whole space-time region.


Fig.2. Graph of the proper distance as function of the coordinate distance in meters for a mass of 1.4 times the mass of Sun. A significant departure of proper distance from the coordinate distance occurs in the region of the Schwarzschild radius.

It is also clear that the new metric approaches the Schwarzschild metric for large distances, so it can be expected that the observational confirmations of GTR, including the light deflection by Sun and Shapiro delay, will also apply here. This can be made more obvious by expressing the proper distance as a function of the coordinate distance. Unfortunately, an approximate analytic formula for large distances is possible only when $r$ is expressed as a function of $\rho$. This becomes:

$$
\begin{equation*}
r(\rho)=\rho \cdot e^{\frac{-R_{s}}{2 \rho}}\left(1-\frac{R_{s}}{2 \rho} \ln \left(\frac{2 \rho}{R_{s}}\right)\right)+\frac{R_{s}}{2} \gamma, \tag{55}
\end{equation*}
$$

where $\gamma$ is the Euler constant $\gamma=.5772156649 \ldots$. For small distances, both the constant and logarithmic terms are omitted. The graph of $\rho$ as function of $r$ can be calculated numerically and the result is shown in Fig.2. The derivation of the light deflection formula and Shapiro delay will be left for future publications. The results of light deflection and Shapiro delay, to the first order of $\mathrm{R}_{\mathrm{s}} / \mathrm{r}$, are identical with the corresponding GTR formulas.

## CONSEQUENCES OF THE NEW METRIC

The new metric allows calculation of several important effects that are directly testable by comparing them with observations. The first one is the gravitational red shift. The recent discovery of the gravitational red shift of the Pulsar EXO 0748-676 ${ }^{2}$ allows an accurate determination of the pulsar mass to radius ratio. The observed value of $Z$ was $Z=0.35$, determined with a high precision. The gravitational red shift formulas are easily obtained directly from the corresponding metrics. For the Schwarzschild metric this becomes:

$$
\begin{equation*}
Z_{s}=\left(1-\frac{R_{s}}{r}\right)^{\frac{-1}{2}}-1 \tag{56}
\end{equation*}
$$

while for the new metric the formula, expressed as function of the coordinate distance, is as follows:

$$
\begin{equation*}
Z_{c}=e^{\frac{R_{s}}{2 \rho(r)}}-1 \tag{57}
\end{equation*}
$$

where the proper distance is evaluated as function the coordinate distance numerically using Eq.55. The plots of the respective red shifts are shown for the neutron star of mass 1.4 times the mass of Sun in Fig.3.

As can be seen from the graphs a reasonable and measurable difference between the predictions of respective theories is obtained. If an independent and accurate determination of the pulsar's radius or mass were found it would be possible to distinguish between these two metrics. It might also be possible to collect statistical data from many pulsars and compare the corresponding statistical distributions of Z shifts. However, for more accuracy, it is necessary to replace the coordinate distance with the apparent distance, since the coordinate as well as the proper distance is not known. As shown in Fig. 2 in the region close to the Schwarzschild radius there is a significant difference between the proper distance and the coordinate
distance. The same is true for the apparent distance, the distance traveled by light, which is directly observable. The coordinate distance can then be calculated from the apparent distance and the plot of Z shift versus the apparent distance made. It is interesting that in such a graph the $Z$ shift saturates at approximately the observed value of Z . This is not shown in this paper.


Fig.3. Gravitational red shift predicted from the Schwarzschild metric and from the new metric as function of the pulsar coordinate radius plotted in meters. The pulsar's mass was assumed to be 1.4 times the mass of Sun. The observed value of Z is shown as a horizontal dashed line.

The speed of light is obtained from the new metric by setting $\mathrm{ds}=0$. For the radial direction this becomes:

$$
\begin{equation*}
\frac{d r}{d t}= \pm c \cdot e^{\frac{-R_{s}}{\rho(r)}} \tag{58}
\end{equation*}
$$

As the photon approaches the origin of coordinates its speed is gradually reduced until the photon finally stops. The oscillating electric and magnetic fields become DC fields. From this result it is also apparent that the particles created in the region within the Schwarzschild radius with very high velocity can escape. This result might represent a possible theoretical support for the observation of particle jets emanating from the centers of many galaxies.

It is also interesting to calculate from the new metric the Newtonian acceleration expressed in coordinate distances. This can be accomplished directly from Eq. 38 when it is rewritten using the new metric. This becomes:

$$
\begin{equation*}
\frac{d^{2} \rho}{d \tau^{2}}=\frac{c^{2}}{2} \cdot \frac{d}{d \rho} e^{B(\rho)} \tag{59}
\end{equation*}
$$

By replacing the proper distance and the proper time with the coordinate distance and the coordinate time the result becomes:

$$
\begin{equation*}
\frac{d^{2} r}{d t^{2}}=\frac{c^{2} B^{\prime}(r)}{2}\left(3 \cdot e^{-3 B(r)}-2 \cdot e^{-2 B(r)}\right) \tag{60}
\end{equation*}
$$

The second term in Eq. 60 represents a repulsive force that dominates at small distances. By substituting for the metric coefficient into Eq. 60 the result becomes:

$$
\begin{equation*}
\frac{d^{2} r}{d t^{2}}=-\frac{c^{2} R_{s}}{2 \rho^{2}(r)} \cdot \frac{3 \cdot e^{\frac{-R_{s}}{\rho(r)}}-2}{e^{\frac{3}{2} \frac{R_{s}}{\rho(r)}}} \tag{61}
\end{equation*}
$$

As r approaches the origin of coordinates the repulsive acceleration approaches to zero. At large distances, $r \gg R_{s}$, the expression reverts back to the standard Newtonian attractive acceleration.

The last phenomenon that will be addressed in this article is the Mercury's perihelion advance. The perihelion advance is one of the more sensitive tests for the metric, since this phenomenon is integrating. The small single orbit perihelion advances accumulate over the long time periods and the resulting accumulation can thus be readily detected. To calculate the perihelion advance from the metric, it is customary to use Lagrangian derived in Eq.54. From the Lagrangian it is then simple to find EL equations of motion and their first integrals. Since the motion of the planet is periodic it is no problem that the Lagrangian in Eq. 54 is expressed in proper distances. The same result would be obtained if the proper distances were transformed to coordinate distances. The first integrals are therefore as follows:

$$
\begin{gather*}
R^{2}(\rho) \cdot \frac{d \varphi}{d \tau}=\alpha  \tag{62}\\
\frac{d t}{d \tau}=e^{B(\rho)}  \tag{63}\\
\left(\frac{d \rho}{d \tau}\right)^{2}=k+c^{2} e^{B(\rho)}-\frac{\alpha^{2}}{R^{2}(\rho)} \tag{64}
\end{gather*}
$$

where $\alpha$ and $k$ are the constants of integration $\left(k=-c^{2}\right)$. It is important to note that Eq. 64 has a universal validity regardless of the coordinate system used. This follows directly from Eq. 38 that also does not depend on the selected coordinates. By using the customary substitution $u=1 / \rho$ and by eliminating $\tau$ from Eq. 62 and 64 , the equations can be reduced to a single equation for $u$ as function of $\varphi$ as follows:

$$
\begin{equation*}
\left(\frac{d u}{d \varphi}\right)^{2} e^{2 R_{s} u}=\frac{k}{\alpha^{2}}+\frac{c^{2}}{\alpha^{2}} e^{R_{s} u}-u^{2} e^{R_{s} u} \tag{65}
\end{equation*}
$$

Dividing Eq. 65 by factor $e^{2 R_{s} u}$, expanding each exponential into the power series keeping only the first two terms, and differentiating the result with respect to $\varphi$, the equation becomes:

$$
\begin{equation*}
\frac{d^{2} u}{d \varphi^{2}}+u=\frac{R_{s} c^{2}}{2 \alpha^{2}}+\frac{3}{2} R_{s} u^{2} . \tag{66}
\end{equation*}
$$

This is the standard form of equation describing the perihelion advance ${ }^{4}$. The advance is calculated to be:

$$
\begin{equation*}
\Delta \varphi \approx \frac{3}{2} \pi \cdot R_{s}\left(\frac{1}{R_{1}}+\frac{1}{R_{2}}\right), \tag{67}
\end{equation*}
$$

where $R_{1}$ and $R_{2}$ are the perihelion and aphelion distances respectively. This formula gives the standard value commonly recognized today for the Mercury's perihelion advance due to gravitationally induced space-time curvature, which is equal to $\Delta \varphi=42.993$ " per century.

## DISCUSSION AND CONCLUSIONS

In this article it was shown that the space-time metric for a central spherically symmetric nonrotating body could be derived from first principles without the use of Einstein's field equation. It was also shown that the new metric agrees well with the Schwarzschild metric for large distances and predicts the same results for the gravitational red shift and Mercury perihelion advance. Larger deviations were found only for the region in the vicinity of Schwarzschild radius. However, the new metric does not predict the existence of black holes, event horizons, and geometry singularity. The new metric allows emissions of photons and relativistic particles from the gravitationally collapsed objects that are typically found in the centers of galaxies.

The derivation of the new metric was developed from simple thought experiments that resulted in new mass equivalence formula with the gravitational mass depending on velocity differently than the inertial mass. It was found that Newton's gravitational law is consistent with the Schwarzschild metric when the new mass equivalence formulas are used. However, it was also concluded that the Schwarzschild metric is not physical and does not correctly describe the space-time in very strong gravitational fields. This is the main reason for such nonphysical pathologies as black holes. The new metric was derived by slightly modifying Newton's gravitational law that was written with proper time and proper distance. The new metric removes one of the principal objections to GTR, which is vanishing of the Riemannian curvature tensor everywhere in the region where the gravitational field resides.

The star model used in the presented work was very simple and more work is needed in the future to further develop this static theory into a complete theory of gravity describing dynamical effects.

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## APENDIX A

It is well known that the Lagrangian for a free falling relativistic particle in the coordinate system that is falling with the particle has the form:

$$
\begin{equation*}
L=c^{2}\left(\frac{d t^{\prime}}{d \tau}\right)^{2}-\left(\frac{d x^{\prime}}{d \tau}\right)^{2} \tag{A1}
\end{equation*}
$$

The primes indicate the co-moving coordinate reference frame. The associated Euler-Lagrange equations yield linear motion without acceleration. The corresponding metric line element is described by the equation:

$$
\begin{equation*}
d s^{2}=c^{2} d \tau^{2}=c^{2} d t^{\prime 2}-d x^{\prime 2} \tag{A2}
\end{equation*}
$$

with the metric determinant $g$ defined as follows:

$$
\left|\begin{array}{l}
g_{t t}, 0  \tag{A3}\\
0, g_{x x}
\end{array}\right|=-c^{2}
$$

Considering first a well-behaved arbitrary coordinate transformation, it is possible to write:

$$
\begin{equation*}
L=c^{2}\left(\frac{\partial t^{\prime}}{\partial t} \frac{d t}{d \tau}+\frac{\partial t^{\prime}}{\partial x} \frac{d x}{d \tau}\right)^{2}-\left(\frac{\partial x^{\prime}}{\partial t} \frac{d t}{d \tau}+\frac{\partial x^{\prime}}{\partial x} \frac{d x}{d \tau}\right)^{2} \tag{A4}
\end{equation*}
$$

After rearrangement this becomes:

$$
\begin{equation*}
L=\left[c^{2}\left(\frac{\partial t^{\prime}}{\partial t}\right)^{2}-\left(\frac{\partial x^{\prime}}{\partial t}\right)^{2}\right]\left(\frac{d t}{d \tau}\right)^{2}-\left[\left(\frac{\partial x^{\prime}}{\partial x}\right)^{2}-c^{2}\left(\frac{\partial t^{\prime}}{\partial x}\right)^{2}\right]\left(\frac{d x}{d \tau}\right)^{2}+2\left[c^{2} \frac{\partial t^{\prime}}{\partial t} \frac{\partial t^{\prime}}{\partial x}-\frac{\partial x^{\prime}}{\partial t} \frac{\partial x^{\prime}}{\partial x}\right] \frac{d t}{d \tau} \frac{d x}{d \tau} . \tag{A5}
\end{equation*}
$$

It is desirable that the coordinate transformation is such that the mixed terms vanish. This requires the following equation to be satisfied:

$$
\begin{equation*}
\left[c^{2} \frac{\partial t^{\prime}}{\partial t} \frac{\partial t^{\prime}}{\partial x}-\frac{\partial x^{\prime}}{\partial t} \frac{\partial x^{\prime}}{\partial x}\right]=0 \tag{A6}
\end{equation*}
$$

It is also desirable that the transformation conserves the value of the determinant g . This implies the following:

$$
\begin{equation*}
-g=c^{2}=c^{2}\left(\frac{\partial t^{\prime}}{\partial t}\right)^{2}\left(\frac{\partial x^{\prime}}{\partial x}\right)^{2}-\left(\frac{\partial x^{\prime}}{\partial t}\right)^{2}\left(\frac{\partial x^{\prime}}{\partial x}\right)^{2}-c^{4}\left(\frac{\partial t^{\prime}}{\partial t}\right)^{2}\left(\frac{\partial t^{\prime}}{\partial x}\right)^{2}+c^{2}\left(\frac{\partial x^{\prime}}{\partial t}\right)^{2}\left(\frac{\partial t^{\prime}}{\partial x}\right)^{2} \tag{A7}
\end{equation*}
$$

Squaring Eq.A6 results in the following:

$$
\begin{equation*}
c^{4}\left(\frac{\partial t^{\prime}}{\partial t}\right)^{2}\left(\frac{\partial t^{\prime}}{\partial x}\right)^{2}+\left(\frac{\partial x^{\prime}}{\partial t}\right)^{2}\left(\frac{\partial x^{\prime}}{\partial x}\right)^{2}-2 \cdot c^{2} \frac{\partial t^{\prime}}{\partial t} \frac{\partial t^{\prime}}{\partial x} \frac{\partial x^{\prime}}{\partial t} \frac{\partial x^{\prime}}{\partial x}=0 \tag{A8}
\end{equation*}
$$

Adding Eq.A7 with Eq.A8 results in:

$$
\begin{equation*}
J^{2}=\left(\frac{D\left(t^{\prime}, x^{\prime}\right)}{D(t, x)}\right)^{2}=\left(\frac{\partial t^{\prime}}{\partial t} \frac{\partial x^{\prime}}{\partial x}-\frac{\partial x^{\prime}}{\partial t} \frac{\partial t^{\prime}}{\partial x}\right)^{2}=1 \tag{A9}
\end{equation*}
$$

which implies that the Jacobian of the transformation equals to unity.
It is now possible to reverse the chain of reasoning and state the following theorem:
Every Lagrangian in the form:

$$
\begin{equation*}
L=e^{A(x, t)}\left(\frac{c d t}{d \tau}\right)^{2}-e^{B(x, t)}\left(\frac{d x}{d \tau}\right)^{2} \tag{A10}
\end{equation*}
$$

with $A(x, t)=-B(x, t)$ can be transformed into a free falling relativistic Lagrangian by a suitable coordinate transformation with the Jacobian equal to unity.


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