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Determination of the Limiting Divergent Infinite Series.

and

A Review of the Divergency of the Harmonic Series.

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Abstract.

This paper investigates the existence of a limiting divergent infinite series and shows this to be the Unitary Series, $\zeta(0)$. This result then necessitates an in depth review of the divergency of the Harmonic Series which suggests that this series may not be divergent.

Contents.

1.0 Introduction.

It is well known that ζ(2), the Basel infinite series was shown to be convergent by Leonard Euler in 1734 with a closed form value of $\pi^2/6$. It is also generally accepted that ζ(1), the Harmonic Infinite series is divergent. The nature of these two series suggests that somewhere in between them lies a limiting divergent series such that all series with equal or progressively larger terms will be divergent, and all series with progressively smaller terms will be convergent.

It is the purpose of this paper to investigate the existence of this limiting divergent series.

2.0 Nomenclature.

The following nomenclature is used in this paper.

ζ(#)	Represents a generic infinite series of the type Zeta.
η(#)	represents a generic infinite series of the type Eta.
u_n	Represents the n^{th} term in the series of interest.
n	Represents the n^{th} value in the series of interest.
m	Represents the m^{th} iteration in the manipulation of the series of interest.
S	Represents a general infinite series sum.
p	Represents a general term in the series of interest.
$f(x)$	Represents a continuous function.
x	Represents the x^{th} value in $f(x)$.

The word 'infinity' and the symbol '∞' represent the concept of an unlimited number of terms in the series of interest, and/or the value tended to by the summation of an unlimited number of terms in the series of interest.

3.0 Determination of the Limiting Divergent Infinite Series.

To begin this process it is necessary to start from a known stable position, ζ(2) has therefore been chosen, viz.

$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\cdots \frac{1}{n^2} + \cdots\cdots + = \frac{\pi^2}{6} \tag{3.1}$$

Because this series is absolutely convergent, it can be arithmetically manipulated provided such manipulation does not affect term order. Therefore writing (3.1) as

$$\zeta(2) = 1 + \frac{1}{4}\left(1 + \frac{4}{9} + \frac{4}{16} + \frac{4}{25} \cdots\cdots \frac{4}{n^2} + \cdots\cdots + \right) = \frac{\pi^2}{6} \tag{3.2}$$

so that

$$1 + \frac{4}{9} + \frac{4}{16} + \frac{4}{25} \cdots\cdots \frac{4}{n^2} + \cdots\cdots + = 4\left(\frac{\pi^2}{6} - 1\right) \tag{3.3}$$

Note that apart from the unity term, each term in (3.3) is larger than the corresponding term in (3.1) and converges to a larger value. Now write (3.3) as

$$1+\frac{4}{9}\left(1+\frac{9}{16}+\frac{9}{25}+\frac{9}{36}\cdots\cdots\frac{9}{n^2}+\cdots\cdots+\right)=4\left(\frac{\pi^2}{6}-1\right) \tag{3.4}$$

so that

$$1+\frac{9}{16}+\frac{9}{25}+\frac{9}{36}\cdots\cdots\frac{9}{n^2}+\cdots\cdots+=9\left\{\frac{\pi^2}{6}-\left(1+\frac{1}{4}\right)\right\} \tag{3.5}$$

The same comment under (3.3) also applies here concerning (3.5) in relation to (3.3). Another iteration gives

$$1+\frac{16}{25}+\frac{16}{36}+\frac{16}{49}\cdots\cdots\frac{16}{n^2}+\cdots\cdots+=16\left\{\frac{\pi^2}{6}-\left(1+\frac{1}{4}+\frac{1}{9}\right)\right\} \tag{3.6}$$

and therefore after *m* iterations

$$1+\frac{(m+1)^2}{(m+2)^2}+\cdots\cdots+\frac{(m+1)^2}{n^2}+\cdots=(m+1)^2\left\{\frac{\pi^2}{6}-\left(1+\frac{1}{4}+\frac{1}{9}+\cdots\frac{1}{m^2}\right)\right\} \tag{3.7}$$

which can be written

$$1+\frac{\left(1+\frac{1}{m}\right)^2}{\left(1+\frac{2}{m}\right)^2}+\cdots\cdots+\frac{m^2\left(1+\frac{1}{m}\right)^2}{n^2}+\cdots=(m+1)^2\left\{\frac{\pi^2}{6}-\left(1+\sum_{p=1}^{m}\frac{1}{p^2}\right)\right\} \tag{3.8}$$

and where while *m* is finite both the LHS and the RHS of (3.8) are clearly finite. Now consider the LHS of (3.8) as both *m* and $n \to \infty$. This gives

$$LHS\Big|_{\substack{m\to\infty \\ n\to\infty}} \Rightarrow 1+1+1+\cdots+\frac{m^2}{n^2}\Big|_{\substack{m\to\infty \\ n\to\infty}} \tag{3.9}$$

The final term in (3.9) appears indeterminant but because both *m* and *n* advance in unitary increments, this ratio cannot be less than nor greater than unity. Consequently, (3.9) approaches the Unitary series as both *m* and $n \to \infty$.

Now consider the RHS of (3.8) as $m \to \infty$.

$$RHS\Big|_{m\to\infty} \Rightarrow \left\{(m+1)^2\left(\frac{\pi^2}{6}-\sum_{p=1}^{m}\frac{1}{p^2}\right)\right\}_{m\to\infty} \tag{3.10}$$

and as *m* increases

$$\sum_{p=1}^{m} \frac{1}{p^2} \Rightarrow \varsigma(2) \tag{3.11}$$

so that (3.10) approaches

$$RHS\big|_{m\to\infty} \Rightarrow \left\{(m+1)^2\left(\frac{\pi^2}{6} - \varsigma(2)\right)\right\}_{m\to\infty} \tag{3.12}$$

which can be interpreted as approaching

$$RHS\big|_{m\to\infty} \Rightarrow \left\{(m+1)^2 \text{ x } (0)\right\}_{m\to\infty} \tag{3.13}$$

Eq.(3.13) also appears indeterminant, but because it is under the limit sign it can be written

$$RHS\big|_{m\to\infty} \Rightarrow \left(\frac{(m+1)^2}{m}\right)_{m\to\infty} \tag{3.14}$$

which reduces to

$$RHS\big|_{m\to\infty} \Rightarrow \left(m + 2 + \frac{1}{m}\right)_{m\to\infty} \Rightarrow \infty \tag{3.15}$$

Thus from (3.9) and (3.15), in accordance with the comment under (3.8), it is proposed that the Unitary series is the limiting divergent series, and all series with equal or progressively larger terms will be divergent. Accordingly, all series with progressively smaller terms will be convergent and possess a finite closed form. This suggests that the Harmonic series may not be divergent but instead convergent with a finite closed form. This is contrary to current thinking. However, this result is not considered surprising as it seems somewhat anomalous in that a series in which all terms are smaller than their predecessor should not be convergent. Therefore, to explore this further, it necessitates a review of the tests that have in the past concluded that the Harmonic series was divergent.

4.0 A Review of the Divergency Tests of the Harmonic Series.

4.1 A Brief Description of the Tests.

There are four main tests that are used to determine the convergence/divergence of any infinite series. They are briefly described below.

(i) The Final Value Test – This states that if $u_n \to 0$ as $n \to \infty$, then the series may be convergent. This is currently considered as a necessary but not sufficient condition.

(ii) The Comparison Test – This states that if each and every term of the series under test is larger than the corresponding term of a known divergent series , then the series under test is divergent. Conversely if the each and every term of the series under test is smaller than a known convergent series, then the series under test is convergent.

(iii) The d'Alembert Ratio Test – This states that if the ratio $u_{(n+1)}/u_n < 1$, then the series under test is convergent. If it is > 1, then the series is divergent, and if it is equal to 1, then the test fails.

(iv) The Cauchy Integral Test – This states that if *f(x)* simulates the general term of the series under test, and if $\int_1^\infty f(x)dx$ exists and is finite, then the series under test is convergent, otherwise it is divergent.

4.2 Application to the Harmonic Series.

Test (i) – The general term of the Harmonic series is

$$u_n = \frac{1}{n} \tag{4.1}$$

and it is clear that as $n \to \infty$, $u_n \to 0$ so that this suggests that the Harmonic series may be convergent. Note however, that if ζ(1) is indeed divergent, because $u_n \to 0$ as $n \to \infty$, then the "final term" has no bearing on the series sum and the divergency would have to be given by the summation of all preceding terms.

Test (ii) – The divergent series that is usually quoted for testing the Harmonic series is

$$S = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots\cdots + \frac{1}{2} + \cdots \tag{4.2}$$

Note that this is

$$S = \frac{1}{2}(1 + 1 + 1 + \cdots\cdots + 1 + \cdots) \tag{4.3}$$

i.e. "one half" of the Unitary series. The test is applied as follows. Writing the Harmonic series as

$$\zeta(1) = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \tag{4.4}$$

and (4.2) as

$$S = \frac{1}{2} + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \tag{4.5}$$

then comparing the bracketed terms in (4.4) to those in (4.5), while those in (4.5) all sum to ½, those in (4.4) all sum to > ½.This suggests that the Harmonic series is indeed divergent. However, if this comparison process is carried on ad infinitum, because of the result in Test (i) above, as n $\rightarrow \infty$, the comparison terms eventually arrive at, firstly for the Harmonic series,

$$\left(u_{(n-p)}+\cdots+u_{(n-3)}+u_{(n-2)}+u_{(n-1)}+u_{(n)}+\cdots+0\right) \tag{4.6}$$

and for the divergent series

$$u_q+u_q+u_q+u_q+u_q+\cdots\cdots+u_q \tag{4.7}$$

where (4.7) still sums to ½, (4.6) may not be > ½. This test is therefore considered inconclusive.

Test (iii) – Applying d'Alembert's test to the Harmonic series, the test ratio is

$$\frac{u_{(n+1)}}{u_n}=\frac{1/(n+1)}{1/n}=\frac{n}{n+1}=\frac{1}{1+1/n} \tag{4.8}$$

and so as n $\rightarrow \infty$, $\frac{u_{(n+1)}}{u_n}\rightarrow 1$ which means the test fails. However, (4.8) $\rightarrow$ 1 from a series of values each < 1 which could be interpreted as a condition of convergence. This is discussed in Appendix A.

Test (iv) – To apply Cauchy's integral test to the Harmonic series, if the general term is $u_n = 1/n$, then a continuous function that simulates this is

$$f(x)=\frac{1}{x} \tag{4.9}$$

and the test is then

$$\int_1^\infty f(x)dx=\int_1^\infty \frac{1}{x}dx=\left[\ln(x)\right]_1^\infty \Rightarrow \infty \tag{4.10}$$

Thus this test also suggests that the Harmonic series is divergent. However, in this case $1/x \rightarrow 0$ as $x \rightarrow \infty$ which is effectively the same as stated above in Test (i) i.e. as $n \rightarrow \infty$ u_n adds nothing to the series sum and therefore in (4.10) the upper limit may give a false result. Therefore this test, in which the integral evaluates to a logarithmic term is also considered inconclusive.
As a corollary note that this test determines the area under the curve $f(x) = 1/x$ between $x = 1$ and $x \rightarrow \infty$. Because $1/x \rightarrow 0$ as $x \rightarrow \infty$, the area under the curve is bounded, i.e. fully enclosed by the value of $f(x)$ when $x = 1$, the curve itself and the x axis. The area must therefore be finite. The area could be said to be infinite due to the fact that the x axis extends to 'infinity', but if the series $\zeta(2)$ is evaluated in this fashion, then the area of $f(x) = 1/x^2$ is bounded in the same manner as above, and therefore the same conclusion must be drawn, i.e. the area is finite because $\zeta(2)$ is known to be convergent.

The problem in applying Cauchy's integral test to $\zeta(1)$ lies in the fact that the integral evaluates to $\ln(x)$which goes to 'infinity' when $x \to \infty$.
However large $\ln(x)$ is at any finite value of x, smaller and smaller amounts of area are being added to the result as x increases. It is therefore considered an anomalous result for the area to suddenly become infinite at the point at which $x \to \infty$ when zero area is added as the limit is reached.
It is therefore considered that this test should not be applicable to any series whose simulated function integrates to a logarithmic term.

4.3 The Relationship Between the Eta and the Zeta Series.

There is a fifth relationship which is also sometimes used to demonstrate the divergence of the Harmonic series. That is the relationship between the Eta and the Zeta infinite series, viz.

$$\zeta(n) = \frac{\eta(n)}{1 - \frac{1}{2^{(n-1)}}} \tag{4.11}$$

and when $n = 1$, (4.11) clearly shows that $\zeta(1) \to \infty$. However, writing (4.11) as

$$\eta(n) = \left(1 - \frac{1}{2^{(n-1)}}\right)\zeta(n) \tag{4.12}$$

so that when $0 < n < 1$, a convergent region of η, this appears to say that multiplying a divergent series by a finite number produces a convergent series. As stated in [2], this cannot be correct and the result obtained from (4.11) must accordingly also be considered suspect because it is a limiting value. Consequently, it is suggested that the relationship of (4.11) should only be valid for $n > 1$.

A corollary to this is shown via an attempt to derive (4.12) when $n = 1$, viz.

$$\eta(1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \cdots \tag{4.13}$$

Add twice the even terms to both sides

$$\eta(1) + 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots\right) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \tag{4.14}$$

so that

$$\eta(1) + \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots\right) = \zeta(1) \tag{4.15}$$

i.e. which appears to say

$$\eta(1) + \zeta(1) = \zeta(1) \tag{4.16}$$

which in turn appears to say that $\eta(1) = 0$ and which is clearly incorrect as it is well known that the closed form of $\eta(1)$ is ln(2).

5.0 Conclusions.

The main problem in discussions as contained herein is that of the necessary reference to 'infinity'. As stated in the Nomenclature, 'infinity' or '∞' is not a number but a term/symbol indicating the concept of an 'unlimited amount'. Yet it is frequently used in mathematics as though it were a number, especially in the application as limits of summations and integrals. However, such reference is sometimes a necessity in analyses and those discussions on subjects such are considered in this paper. Although the discussions here have avoided this errant use of 'infinity' as much as possible, i.e. by referring to terms as approaching infinity, in order to provide clarity and advance the arguments, it has still in some cases been necessary to refer to the 'final term' of an infinite series when of course such a term does not exist.
Consequently, the identification of the limiting divergent series as the Unitary series and the ensuing dissertation on tests as applied to the Harmonic series, cannot be said to prove that the Harmonic series is not divergent. Instead it is proposed that the convergence/divergence of this series should still perhaps be considered an open question. A question that can only be answered when and if a sound analytical proof one way or the other is devised. However, although such an analytical proof is not yet available, the problem can still be investigated by a semi-analytical/empirical approach.

Finally, as a consequence of the possible acceptance of the results obtained herein, and should the Harmonic series subsequently be confirmed as convergent, it is proposed that Test (i), the Final Value Test, should be considered as a sufficient as well as a necessary test.

Appendix A

Further Comments Regarding the d'Alembert Ratio Test.

Subsequent to the application of this test to the Harmonic series it was stated that if $u_{(n+1)}/u_n \to 1$ from a series of values < 1, this result could be interpreted as a condition of convergence. As a corollary consider the following series

$$S = 1 + 2 + 3 + 4 + \cdots + n + \cdots \tag{A.1}$$

Applying d'Alembert's test gives

$$\frac{u_{(n+1)}}{u_n} = \frac{n+1}{n} = 1 + \frac{1}{n} \tag{A.2}$$

and obviously as $n \to \infty$, $u_{(n+1)}/u_n \to 1$, indicating that the test fails, but clearly (A.1) is a divergent series as all other tests confirm. In this case $u_{(n+1)}/u_n \to 1$ from a series of values > 1 and therefore this result should be interpreted as a condition of divergence. The converse as stated above should therefore also be considered valid.

References.

[1] J.A.Green, *Sequences and Series, (Library of Mathematics, Editor, Walter Ledermann),* Routledge and Kegan Paul, London, 1958.

[2] P.G.Bass, *The Bernhard Riemann Hypothesis – A Resolution,* www.relativitydomains.com.