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New Insight in Noether's Theorem

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Abstract:

We discuss Noether's theorem [1-7] confined to mechanical systems and show that a symmetry of a system must already be a symmetry of the space. This provides a framework within which one may seek conserved quantities pertaining to a system out of those pertaining to the space. The symmetries of a system are dictated by its potential energy. It is shown that a momentum operator arising from an infinitesimal motion of the space, or a Killing vector field, is conserved if the directional derivative of the potential energy by this Killing vector field vanishes.

1.Introduction

We view the physical space as possessing a Euclidean geometry structure, which allows to refer the space to a rectangular Cartesian frame $S \equiv ox_1x_2x_3$ whose origin o can be chosen at any point of space, and whose axes' directions are at our will. The space is endowed naturally by an inner product by which the distance between each two points is the Euclidean norm of the displacement vector, and the angle between two vectors is determined through the inner product. The arbitrariness in choosing the origin and the directions of the coordinate axes are equivalent to say that the process of translating the space as a whole by an arbitrary vector as well as the process of rotating the space about an arbitrary axis by an arbitrary angle preserve the length of the displacement vector between each two points and the angle between them. Because the space looks geometrically equivalent to itself after translation or rotation, the latter imaginary processes are symmetry transformations of the space.

A symmetry transformation, or a motion, of the space (or a manifold) preserves the metric of the space; it maps the space isometrically on itself [8-13] with the distance between each two points remaining unchanged. Continuous symmetry transformations of the 3-Euclidean space E_3

$$\bar{x}^k = f^k(x_1, x_2, x_3; \alpha_1, \dots, \alpha_6), \qquad (k = 1,2,3) \tag{1.1}$$

consist of rotations and translations, and it forms a Lie group with six essential parameters $(\alpha_1, \dots, \alpha_6 \in R)$. The group of isometries of the space (1.1), also called the group of motions of E_3 [10,11] , gives rise to 6 linearly independent metric preserving vector fields, or Killing fields, X_μ ($\mu = 1,2, \dots ,6$), which are its generators [10]. The set $\{X_\mu\}$ is a basis for the Lie algebra LA associated with the Lie group (1.1). Every element of LA can be expressed as a linear combination in the elements of $\{X_\mu\}$, and LA is closed under taking linear combinations of any number of elements, as well as, taking the commutator of any two elements. LA consists of all Killing vectors of the space; all infinitesimal generators of rotations and translations as well as any linear combinations in them. The basis $\{X_\mu\}$ can be chosen as three infinitesimal generators of rotations about the coordinates axes and three infinitesimal generators of translations along them. On fixing five parameters in (1.1) one obtains an one-parameter group of symmetries of the space. In rectangular Cartesian coordinates it is straight forward to list six one-parameter groups of symmetries, or motions, whose generators are (apart from multiplicative constants) the components of momentum on the three coordinate axes as well as the components of the angular momentum about them. Finding the space's motions when curvilinear coordinates are employed in E^3, or when we consider Riemannian spaces or manifolds, can be quite difficult, and it is easier in this case to start from the equations of the infinitesimal generator of motion of the space. The Killing equations [10,11]

$$\eta_k g_{ij,k} + \eta_{k,j} g_{kj} + \eta_{k,j} g_{ik} = 0 \quad (i,j = 1,2,3) \tag{1.2}$$

determine the Killing vector fields or the infinitesimal generators of the space's motions, $X = \eta_i\, \partial/\partial x_i$, if they exist. In the Killing equations (1.2), g_{ij} are the covariant components of the metric tensor, comma denotes differentiation with respect to the variable that follows it, and sum is implied of course on the repeated index k, Summation on repeated indices is used throughout this work.

Before we discuss the symmetries of a mechanical system and present a revised statement of Noether theorem we brief some relevant materials which include variational methods, Euler-Lagrange equations, the Hamiltonian formalism, and Noether theorem.

2. Variational Method, Lagrange's Equation, and Hamiltonian Formalism

Let $\mathcal{L}(q_1, \dots, q_s, \dot{q}_1, \dots, \dot{q}_s, t)$ be a function in the variables q_i, their derivatives $\dot{q}_i = dq/dt$, and the parameter t. The variational technique [14,15] determines the curve $q_i = q_i(t), (i = 1, \dots, s)$, that connects the points $q_i(t_1)$ and $q_i(t_2)$, along which the integral

$$S = \int_{t_1}^{t_2} \mathcal{L}\, dt \tag{2.1}$$

attains a stationary value. The curve $q_i(t) + \delta q_i(t), (i = 1.2 \ldots., s)$, where $\delta q_i(t)$ are infinitesimal increments is infinitesimally close to the curve $q_i(t)$ along which S is stationary. We remind the reader that, if $f(x)$ is a function with a stationary value at x_0 then its increment $\delta f = f(x_0 + \delta x) - f(x_0) \approx \left.\frac{df}{dx}\right|_{x_0} \delta x$ vanishes in an infinitesimal neighborhood of x_0. Similarly, because S has a stationary value on the curve $q_i(t)$,

$$\delta S = \delta \int_{t_1}^{t_2} \mathcal{L}\, dt = \int_{t_1}^{t_2} \delta\mathcal{L}\, dt = \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q_i}\delta q_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta \dot{q}_i\right) dt = 0. \qquad (2.2)$$

We appeal here to a basic rule in the calculus of variation; that is

$$\frac{d}{dt}\delta q_i = \delta \frac{dq_i}{dt} = \delta \dot{q}_i. \qquad (2.3)$$

Indeed, the perturbed coordinates $q_i + \delta q_i$ and their derivatives $\dot{q}_i + \delta \dot{q}_i$ on the adjacent curve are related by

$$\dot{q}_i + \delta \dot{q}_i = \frac{d}{dt}(q_i + \delta q_i) = \dot{q}_i + \frac{d}{dt}\delta q_i \rightarrow \delta \dot{q}_i = \frac{d}{dt}\delta q_i. \qquad (2.4)$$

Now we make use of equation (2.3) and integrate the second term in the integrand by parts:

$$\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta q_i\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\delta q_i dt = 0 \qquad (2.5)$$

Because the δq_i vanish at the end points of the curve we have $\delta q_i(t_1) = \delta q_i(t_2) = 0$, and the first term is null. Focusing now on the integral which vanishes for arbitrary δq_i and taking all δq_i except one at a time equal to zero, yields Euler equations

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0 \qquad (i = 1,2, \ldots., s). \qquad (2.6)$$

The function $\mathcal{L}$ in Euler's equations is already given, while the curve $q_i = q_i(t)$ on which S is stationary is determined.

The Lagrange Equation: Euler's equations single out the path between $q_i(t_1)$ and $q_i(t_2)$ along which the integral (2.1) with respect to the path's parameter is stationary. The resulting path $q_i = q_i(t)$ changes naturally with $\mathcal{L}$.

To determine the motion of a particle with a given potential energy, Lagrange adopted in Euler's equations a specific function $\mathcal{L}$, called the Lagrangian, whose variables are the generalized coordinates q_i, the generalized velocities $\dot{q}_i$, and possibly time *t*. To determine the Lagrangian $\mathcal{L}$, we start by Newton equation of motion of a particle in a force field derived from the potential energy $V(x_1, x_2, x_3, t)$

$$\frac{d}{dt} m\boldsymbol{v} = -\nabla V, \tag{2.7}$$

and proceed to write it in the form (2.6). In a Cartesian system of coordinate (x_1, x_2, x_3) we have

$$\frac{d}{dt} m\dot{x}_i + \frac{\partial V}{\partial x_i} = 0 \; or \; \frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i} + \frac{\partial V}{\partial x_i} = 0 \quad (i = 1,2,3) \tag{2.8}$$

where $T = \frac{1}{2} m\dot{x}_i\dot{x}_i$ is the kinetic energy of the particle. Defining Lagrange function by $\mathcal{L} = T - V$, and making the substitution $\frac{\partial T}{\partial \dot{x}_i} = \frac{\partial \mathcal{L}}{\partial \dot{x}_i}$ and $\frac{\partial V}{\partial x_i} = -\frac{\partial \mathcal{L}}{\partial x_i}$ in (2.8) we obtain Lagrange equations of motion (2.6). The Lagrange equations, so obtained, are evidently equivalent to Newton's second law of motion,

The *generalized momenta* are defined by

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = p_i(q, \dot{q}, t) \;\; (i = 1,2, \dots, s) \tag{2.9}$$

where q and $\dot{q}$ stand collectively for the generalized coordinates and generalized velocities respectively. By (2.9) the Lagrange's equations (2.6), which describes the motion of *n* particle with *s* degrees of freedom in the potential field $V(q_1, \dots, q_s, t)$, can be written in the form

$$\dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i} = f_i \quad (i = 1, \dots, s) \tag{2.10}$$

The quantities $f_i = \partial \mathcal{L}/\partial q_i$ are called the *generalized forces* in the directions of the generalized coordinates q_i.

The Hamiltonian Formalism [14,15]: The Hamiltonian function in q_i and p_i

$(i = 1, \dots, s)$ is defined by

$$H(q, p, t) = p_i\dot{q}_i - \mathcal{L} \tag{2.11}$$

Taking the differential of both sides yields

$$\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt = \dot{q}_i dp_i + p_i d\dot{q}_i - \frac{\partial \mathcal{L}}{\partial q_i} dq_i - \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial \mathcal{L}}{\partial t} dt$$

$$= \dot{q}_i dp_i - \frac{\partial \mathcal{L}}{\partial q_i} dq_i - \frac{\partial \mathcal{L}}{\partial t} dt = \dot{q}_i dp_i - \dot{p} dq_i - \frac{\partial \mathcal{L}}{\partial t} dt$$

Equating the coefficients of the differentials on both sides yields

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \ \dot{p} = -\frac{\partial H}{\partial q_i}, \ \frac{\partial H}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} \quad (i = 1,2, \dots, s) \tag{2.12}$$

The first two equations in (2.12) are Hamilton's equations of motion. We shall assume that the potential energy is a function of the generalized coordinates and possibly of time too, $V = V(q,t)$, and that the kinetic energy is a quadratic form in the generalized velocities, $T = \frac{1}{2} m g_{ij} \dot{q}_i \dot{q}_j$, where g_{ij} is the space's metric. The first term in (2.11) is

$$\dot{q}_i p_i = \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \dot{q}_i \frac{\partial}{\partial \dot{q}_i} \left(\frac{1}{2} m g_{kj} \dot{q}_k \dot{q}_j \right) = m g_{kj} \dot{q}_k \dot{q}_j = 2T \ . \tag{2.13}$$

A well-known theorem regarding differentiation of homogeneous functions of the second rank may also be used to obtain the latter result. Substituting in (2.11) yields

$$H = 2T - \mathcal{L} = T + V \ . \tag{2.14}$$

To obtain H, as stated in (2.11), a function in q and p we express the kinetic energy in terms of the generalized momenta p_i through expressing the generalized velocities $\dot{q}_i$ in terms of p_i:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i} \left(\frac{1}{2} m g_{kj} \dot{q}_k \dot{q}_j \right) = m g_{ij} \dot{q}_j$$

$$\rightarrow g^{ki} p_i = g^{ki} \left(m g_{ij} \dot{q}_j \right) = m \dot{q}_k \tag{2.15}$$

Substituting in the expression of the kinetic energy we get

$$T = \frac{1}{2} m g_{ij} \dot{q}_i \dot{q}_j = \frac{1}{2} m g_{ij} \left(\frac{1}{m} g^{ki} p_k \right) \left(\frac{1}{m} g^{rj} p_r \right) = \frac{1}{2m} g^{rk} p_r p_k \tag{2.16}$$

where g^{rk} is the contravariant metric tensor.

A physical observable is any function (differentiable) of the form $F(q,p,t)$. The rate of change of F, or its equation of motion, is

$$\frac{dF}{dt} = \frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial p_i} \dot{p}_i + \frac{\partial F}{\partial t} = \frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i} + \frac{\partial F}{\partial t}$$

$$\equiv \{F,H\} + \frac{\partial F}{\partial t} \tag{2.17}$$

The quantity $\{F,H\}$ is called the Poisson Bracket of the observable F with the Hamiltonian H. If F does not depend on time explicitly then $\frac{dF}{dt} = \{F,H\}$, and F is a constant of motion if and only if its Poisson bracket with H vanishes. However, F cannot be a constant of motion if it depends explicitly on time, although its Poisson bracket with H is zero. Taking in particular $F = H$, we have by (2.17)

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = \frac{\partial V}{\partial t} \tag{2.18}$$

It follows that the total energy $H = T + V$ is a constant of motion if and only if the potential energy does not depend on time explicitly. Since

The generalized coordinates and generalized momenta form coordination (q,p) of the phase space; they satisfy the Poisson brackets relations

$$\{q_i, p_j\} = \delta_{ij},\ \{q_i, q_j\} = 0,\ \{p_i, p_j\} = 0 \tag{2.19}$$

A generalized coordinate q_i and a generalized momentum p_i are described as canonically conjugate; they have the Poisson bracket $\{q_i, p_i\} = 1$.

3. Noether Theorem:

A coordinate q_i that does not appear in $\mathcal{L}$ is called ignorable (or cyclic); it signifies the absence of the generalized force $f_i = \partial\mathcal{L}/\partial q_i$in its direction. As a result, the conjugate momentum p_i does not change its value in the course of time, i.e. it is conserved ($p_i = const$). The absence of q_i indicates a symmetry in the system with respect to this coordinate. Noether's theorem generalizes the latter observation to include a broader type of symmetries. In Noether theorem, a symmetry means a transformation of the generalized coordinates, generalized velocities, and possibly of the time, that leaves the Lagrangian unchanged.

Statement of the Noether Theorem [1-7]: If the transformations

$$q_i(t) \to q_i(t) + \epsilon\eta_i(t), \dot{q}_i(t) \to \dot{q}_i(t) + \epsilon\dot{\eta}_i(t), t \to t \quad (i = 1, \dots, s) \tag{3.1}$$

where ϵ is a small number, is a symmetry transformations for some functions $\eta_i(t)$, then the quantity

$$p_i\eta_i(t) \ \text{(sum on } i\text{)} \tag{3.2}$$

is a constant of motion, i.e., it is conserved.

Proof: The Lagrangian is invariant under (3.1) when it is a symmetry transformation:

$$\delta\mathcal{L} = \mathcal{L}(q_i(t) + \epsilon\eta_i(t), \dot{q}_i(t) + \epsilon\dot{\eta}_i(t), t) - \mathcal{L}(q_i(t), \dot{q}_i(t), t) = 0 \quad (3.3)$$

The following straightforward sequence of equivalent equations, with the first resulting from (3.3), lead to the required result

$$\frac{\partial\mathcal{L}}{\partial q_i}\gamma_i(t) + \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\dot{\eta}_i(t) = 0 \leftrightarrow \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\eta_i(t) + \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\dot{\eta}_i(t)) = 0 \ \text{(by(2.6))}$$

$$\leftrightarrow \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\eta_i(t)\right) = 0 \leftrightarrow \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\eta_i(t) \ \text{is conserved} \leftrightarrow \ p_i\eta_i(t) \ \text{is conserved.}$$

Still a more direct way to prove Noether theorem is seen from equation (2.5). Since the motion obeys Euler Lagrange equation (2.6), the second term in (2.5) vanishes, giving

$$\delta S = \left[\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\delta q_i(t)\right]_{t_1}^{t}, \quad (3.4)$$

where we took the end point at an arbitrary instant of time t. We will not assume that δq_i vanishes at the end points, and hence the change of the action resulting from the variation (3.1) is given by (3.4). If however, the transformation (3.1) is a symmetry transformation then $\mathcal{L}$ does not change by the transformation and $\delta S = 0$, which proves Noether theorem.

Noether's theorem is often stated as follows: Whenever we have a continuous symmetry of $\mathcal{L}$ there will be an associated conservation law.

4. Continuous System's Symmetry

We note first that a system's symmetry is related to the type of force's field exerted on the system, and has nothing to do with its shape. A coordinate transformation

$$\bar{x}_i = f_i(x_1, x_2, x_3; \alpha), \ or \ \bar{\boldsymbol{r}} = \boldsymbol{f}(\boldsymbol{r}, \alpha) \quad (4.1)$$

where α is a real parameter, induces a unitary transformation $U_{\boldsymbol{f}}(\alpha)$ in the Hilbert space [16-18] of absolutely square integrable functions, $L^2(E_3)$; it is defined by

$$\left(U_f\psi\right)(\bar{r}) \equiv \bar{\psi}(\bar{r}) = \psi(r), \ \ \psi \in L^2(E_3) \quad (4.2)$$

If the family of coordinate transformations (4.1) with $\alpha \in \mathbb{R}$ forms an one-parameter group then so does the corresponding family of unitary transformations $\{U_f(\alpha): \alpha \in \mathbb{R}\}$. The generator of the latter group [19-20]

$$X = \frac{\partial f_k}{\partial \alpha}\bigg|_{\alpha=0} \frac{\partial}{\partial x_k} \equiv \eta_k(x)\ \frac{\partial}{\partial x_k} \quad (\text{sum on } k) \tag{4.3}$$

is a complete vector field. There corresponds to the vector field X [16,21] a classical momentum $P = \eta_k(x)\, p_k$, where p_k is the generalized momentum conjugate to x_k (in the current case $p_k = m\dot{x}_k$), and the quantum momentum

$$\widehat{\mathrm{P}} = -i\hbar(X + \tfrac{1}{2} div\ X) \tag{4.4}$$

which is essentially self-adjoint on the domain [16-18]

$$D_p = \{f : f \in C^1(E_3), f, \widehat{\mathrm{P}}f \in L^2(E_3)\}, \tag{4.5}$$

where $C^1(E_3)$ is the set of continuously differentiable functions on the space.

We shall confine our attention to mechanical aspects of a physical system, which can be an electron, a molecule, a pendulum, a planet, etc…. . We may imagine moving a physical system with respect to a coordinate frame S and thus changing, in no time, its position and orientation. The new hypothetical configuration of the system can be achieved by combining a translation by some vector $\boldsymbol{b}$ and a rotation about a some axis by an angle φ (the active view). An equivalent way to give the system the new configuration is to refer it to a new coordinate frame $\bar{\mathrm{S}}$ that results from S through a rotation by an angle $-\varphi$ about some axis followed by a translation by a vector $-\boldsymbol{b}$. i.e. to perform a suitable coordinate transformation $\bar{\boldsymbol{r}} = R\boldsymbol{r} + \boldsymbol{b}$, where R is the rotation matrix and $\boldsymbol{b}$ is a translation vector (the passive view).

A physical observable, or an operator A in $L^2(E_3)$, is transformed under the coordinate transformation (4.1) to the operator $\bar{A}(\boldsymbol{r}) = U_f A(\boldsymbol{r}) U_f^{-1} = A(f^{-1}\boldsymbol{r})$ [22,23]. The observable A is said to be invariant under the transformation (4.1) if $\bar{A}(\boldsymbol{r}) = A(\boldsymbol{r})$, which is equivalent to A commuting with U_f, $[A, U_f] = 0$.

A physical system posses a symmetry transformation, $\bar{\boldsymbol{r}} = \boldsymbol{f}(\boldsymbol{r}, \alpha)$, if the system and its effective environment (relevant surroundings) are indistinguishable from themselves before and after the transformation. This implies that the transformation must preserve the metric as well as the system’s potential energy, which means that the geometry of the surroundings and the prevailing forces are invariant under the transformation. The latter statements lead to an important conclusion: the transformation (4.1) is a symmetry transformation of a physical system if and only if its kinetic energy T and potential energy V are separately invariant under the transformation. Equivalently,

$$[T, U_f] = 0, [V, U_f] = 0 \tag{4.6}$$

This does not mean of course that either T or V is conserved. In general neither T nor V is a constant of motion although their sum is, when V does not depend on time explicitly. The requirements (4.6) are more stringent than the commonly accepted fact that the transformation (4.1) is a symmetry transformation of the physical system if

$$[H, U_f] = 0 \tag{4.7}$$

The latter equation results of course by summing the equations (4.6).

A direct consequence from our conclusion (4.6) is that: Confining our consideration to system's motion, *the set of spatial continuous symmetries of any physical system is a subset of the symmetries of the space; and it consists therefore of translations, rotations, or a composition of them.*

The generator $X = \eta_k(x)\frac{\partial}{\partial x_k}$ of the 1PG (4.1), when the latter is a symmetry transformation, preserves the metric, and in particular the volume element. It follows that $div\, X = 0$, which may also be proved directly. Indeed, X here satisfies the Killing equations (1.2), which on multiplying by g^{ij} yields

$$\eta_k g^{ij}\, g_{ij,k} + 2\eta_{k,k} = 0 \ \rightarrow \frac{1}{2g} g_{,k}\eta_k + \eta_{k,k} = 0 \rightarrow \left(\sqrt{g}\eta_k\right)_{,k} = 0 \rightarrow$$

$$div\, X \equiv \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_k}\left(\sqrt{g}\eta_k\right) = 0,$$

where g is the determinant of the covariant metric tensor g_{ij}, and comma denotes differentiating with respect to the index following it.

Reverting to main stream, there corresponds to the Killing field $X = \eta_k(x)\,\partial/\partial x_k$, the classical momentum $P = \eta_k p_k$ and the quantum momentum [16-18]

$$\hat{P} = -i\hbar\eta_k(x)\ \frac{\partial}{\partial x_k} \tag{4.8}$$

which is essentially self-adjoint on the domain (4.5).

The one-parameter unitary group U_f acting in L^2 and generated by X can be represented by the exponential function [19,20,24]

$$U_f = e^{\alpha X} \quad (\alpha \in \mathbb{R}) \tag{4.10}$$

In vicinity of the identity, $\alpha = 0$, we have

$$U_f = I + \alpha X + 0(\alpha^2) = S_\alpha + 0(\alpha^2) \tag{4.11}$$

where $S_\alpha = I + \alpha X$ is an infinitesimal symmetry transformation. The expression (4.11) of U_f in vicinity of the identity operator I shows that the potential energy commute with U_f if and only if it commutes with its generator X:

$$[V, U_f] = 0 \leftrightarrow [V, S_\alpha] = 0 \leftrightarrow [V, X] = 0 \tag{4.11}$$

Parallel relations holds for the kinetic energy T and the Hamiltonian H; they commute with U_f if and only if they commutes with X.

It is well known that the kinetic energy is invariant under rotations and translation, and hence commute with X. It follows there that (4.1) is a symmetry transformation of a physical system if and only if X commute with the system potential energy. Noting that $[X, V] = 0 \leftrightarrow X(V) = 0$, we state:

Theorem: The quantum (classical) momentum observable $\mathrm{P} = -i\hbar\eta_k(x)\,\frac{\partial}{\partial x_k}$ $(\eta_k p_k)$ is a constant of the motion if and only if the Lie derivative of the potential energy by the vector field X vanishes:

$$\eta_k(x)\frac{\partial}{\partial x_k}V = 0. \tag{4.12}$$

The latter relation can be put in the form $X.\nabla V = 0$, which signifies that the vector field is perpendicular to gradient V (to the force field), and hence, is in the level surfaces of V. We should keep in mind however that X has to be already a generator of symmetry of the space.

5. Examples

Example 1. Consider a particle with the Lagrangian

$$\mathcal{L} = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - V(a_1x_1 + a_2x_2 + a_3x_3) \tag{5.1}$$

Here the potential is constant on a plane Π, determined by $a_1x_1 + a_2x_2 + a_3x_3 = k$, and changes its value from a plane k to another k'. The symmetries here consist of:

(1) The group of translations $x_i \to x_i + b_i\ (i = 1,2,3)$ in the plane Π. The components of the displacement vector $\boldsymbol{b} = (b_1, b_2, b_3)$ are not independent because it is in the plane Π. Indeed, in order V remains unchanged by the transformation, i.e., is an invariant, we should have $\sum_{i=1}^{3} a_i(x_i + b_i) = k$, which yields $\sum_{i=1}^{3} a_i b_i = 0$, or as to say $\boldsymbol{n}.\boldsymbol{b} = 0$.

(2) Rotations in the plane Π about an axis $\boldsymbol{n} = (a_1, a_2, a_3)$ passing through any point.

Before checking the efficiency of the formula (4.12) we remind the reader that a generator of symmetry can only be linear combinations of $\partial_i\ and\ \frac{\boldsymbol{i}}{\hbar}L_i$ and only those combination that satisfy (4.12) are the possible generators of continuous spatial symmetries. It is to be noted that the results we obtain hold equally for classical and quantum momenta.

(i) Two independent generators of the group of translation in Π can be chosen as

$$X_1 = a_2\,\partial/\partial x_1 - a_1\,\partial/\partial x_2\,, X_2 = a_3\,\partial/\partial x_2 - a_2\,\partial/\partial x_3\,.$$

Any other generator of this group is a linear combination in X_1 and X_2. It is apparent that the Lie derivative of V by the X_1 and X_2 vanishes, which give rise to the conserved momenta $P_1 = a_2p_1 - a_1p_2$ and $P_2 = a_3p_2 - a_2p_3$. We may replace X_2 by $X_2^{'} = -a_1a_3\partial_1 - a_2a_3\partial_2 + (a_1^2 + a_2^2)\partial_3$ which is orthogonal to X_1.

(2) The generator of the group of rotations is $X_3 = \frac{\boldsymbol{i}}{\hbar}\boldsymbol{n}.\boldsymbol{L} = \frac{\boldsymbol{i}}{\hbar}a_iL_i = \frac{\boldsymbol{i}}{\hbar}P_3$, which is the component of the angular momentum vector $\boldsymbol{L} = (L_1, L_2, L_3)$ on $\boldsymbol{n}$.

Any other conserved momentum is a linear combination of P_1, P_2 and P_3. It is noted that all arguments and results apply to classical and quantum momenta.

Let's compare our theory with the familiar one, and verify that the derived momenta are constants of the motion through checking their commutation relations with the Hamiltonian

$$H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + V(a_1x_1 + a_2x_2 + a_3x_3)$$

For P_1, for instance, we have

In the quantum case: $[P_1, H] = [P_1, V] = -i\hbar[X_1, V] = -i\hbar X_1(V) = 0$

In the classical case:

$$\{P_1, H\} = \frac{\partial P_1}{\partial x_k}\frac{\partial H}{\partial p_k} - \frac{\partial P_1}{\partial p_k}\frac{\partial H}{\partial x_k} = -\frac{\partial P_1}{\partial p_k}\frac{\partial H}{\partial x_k} = -\left(a_2\frac{dV}{dX}a_1 - a_1\frac{dV}{dX}a_2\right) = 0$$

To apply Noether Treatment to the given Lagrangian we set an 1-parameter group of transformations that leaves the Lagrangian $\mathcal{L}$ unchanged. The coordinates transformations

$$(\bar{x}_1, \bar{x}_2, \bar{x}_3) \to \left(x_1 + \frac{\alpha}{a_1}, x_2 + \frac{\alpha}{a_2}, x_3 - \frac{2\alpha}{a_3}\right)$$

form an 1-parameter group of transformations (1PG) that preserve $\mathcal{L}$. By Noether theorem, the quantity

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_k}\frac{d\bar{x}_k}{d\alpha} = m\left(\frac{\dot{x}_1}{a_1} + \frac{\dot{x}_2}{a_2} - \frac{2\dot{x}_3}{a_3}\right)$$

is an invariant which we write in the form

$$P_3 = a_2a_3p_1 + a_1a_3p_2 - 2a_1a_2p_3$$

Also the 1PG

$$(\bar{x}_1, \bar{x}_2, \bar{x}_3) \to (x_1 - 2\alpha/a_1\,, x_2 + \alpha/a_2\,, x_3 + \alpha/a_3)$$

preserves $\mathcal{L}$ and leads to the invariant

$$P_4 = -2a_2a_3p_1 + a_1a_3p_2 + a_1a_2p_3$$

It clear of course that P_3 and P_4 are linear combinations in P_1 and P_2.

Example 2. We apply (4.12) to the following example which is given in [7]. Consider a one particle system with the Lagrangian given in cylindrical coordinates by

$$\mathcal{L} = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\emptyset}^2 + \dot{z}^2\right) - V(\rho, a\emptyset + z)$$

It is clear that Lie derivative of V by the vector field $X = \frac{\partial}{\partial\emptyset} - a\frac{\partial}{\partial z}$ vanishes. X is an infinitesimal motion of the space because it is a linear combination of the infinitesimal motions of $\frac{\partial}{\partial\emptyset}$ and $\frac{\partial}{\partial z}$. Since $X(V) = 0$, the momentum $P = p_\emptyset - ap_z$ is conserved.

Conclusion

The set of symmetries of a mechanical system is a subset of the symmetries of the space. A space's symmetry is admitted as a symmetry of the system if the directional derivative of the potential energy by its infinitesimal generator vanishes. The found results provide a scheme to specify the sought symmetries, and shed new insight in the inspiring and beautiful Noether's theory.

References

[1] J. Butterfield, On Symmetry and Conserved Quantities in Classical Mechanics, arXiv.physics/0507192v1 [physics.class-phys] 27 July 2005

[2] V. Ilisie, Concepts in Quantum Field Theory. UNITEXT for physics, DOI 10.1007/978-3-319-22966-9-2, Springer International Publishing Switzerland 2016.

[3] G4003 – Advanced Mechanics
The Noether theorem - Columbia Physics
phys.columbia.edu/~nicolis/NewFiles/Noether_theorem.pdf

[5] Noether's Theorem - MathPages
www.mathpages.com/home/kmath564/kmath564.htm

[6] Noether's Theorem
Noether's Theorem in Classical Mechanics revisited, Rubens M. Marinho Jr., arXiv: Physics/0608264v1, 26 Aug 2006

[7]Noether's Theorem
courses.physics.ucsd.edu/2010/Fall/physics110a/LECTURES/CH07.pdf

[8] Note on Symmetry and Constants of motion
www.uio.no/studier/emner/matnat/fys/FYS3120/v04/.../Symmetry.pdf

[9] Continuous Symmetries and Conservation Laws
users.physik.fu-berlin.de/~kleinert/b6/psfiles/Chapter-7-conslaw.pdf

[10] Eisenhart L P 1964 *Riemannian Geometry* (Princeton: Princeton University Press)

[11] Matsushima Y 1972 *Diflerentiable Manifolds* (New York: Dekker) p 91

[12] Brickell F and Clark R **S** 1970 *Differentiable Manifolds* (New York: Van Nostrand Reinhold)

[13] Synge J L and Schild **A** 1966 *Tensor Calculus* (Toronto: University of Toronto Press)

[14] John R Taylor (2004) Classical Mechanics, University Science Books, Sausalito, California

[15] Lecture Notes on Classical Mechanics for Physics 106ab Sunil .Galwala..
www.astro.caltech.edu/~golwala/ph106ab/ph106ab_notes.pdf

[16] Mackey G W 2004 *The Mathematical Foundations of Quantum Mechanics* Dover Publications

[17] Wan K K and Viazminsky C 1977 *Prog. Theor. Phys.* **58** 1030

[18] Kay-Kong Wan and C Viazminsky, 1979, Journal of Physics A: Mathematical and General, Volume 12, Number 5

[19] Don Bernett Lichtenberg, 1978, Unitary Symmetry and Elementary Particles, Academic Press

[20] B. G. Wybourne (1975) Classical Groups for Physicists., John Wiley & Sons, London 1974

[21] Loomis L H and Sternberg **S** 1990 *Advanced Calculus* (Reading, MA: Addison-Wesley)

[22] Viazminsky C. (2005) Quantum Mechanics, Aleppo University Publications.

[23] Cornwell J F (1997) Group Theory in Physics. Academic Press.

[24] Gallier J , 2001 Basics of Classical Lie Groups: The Exponential Map, Lie Groups, and Lie Algebras. In: Geometric Methods and Applications. Texts in Applied Mathematics, vol 38. Springer, New York, NY