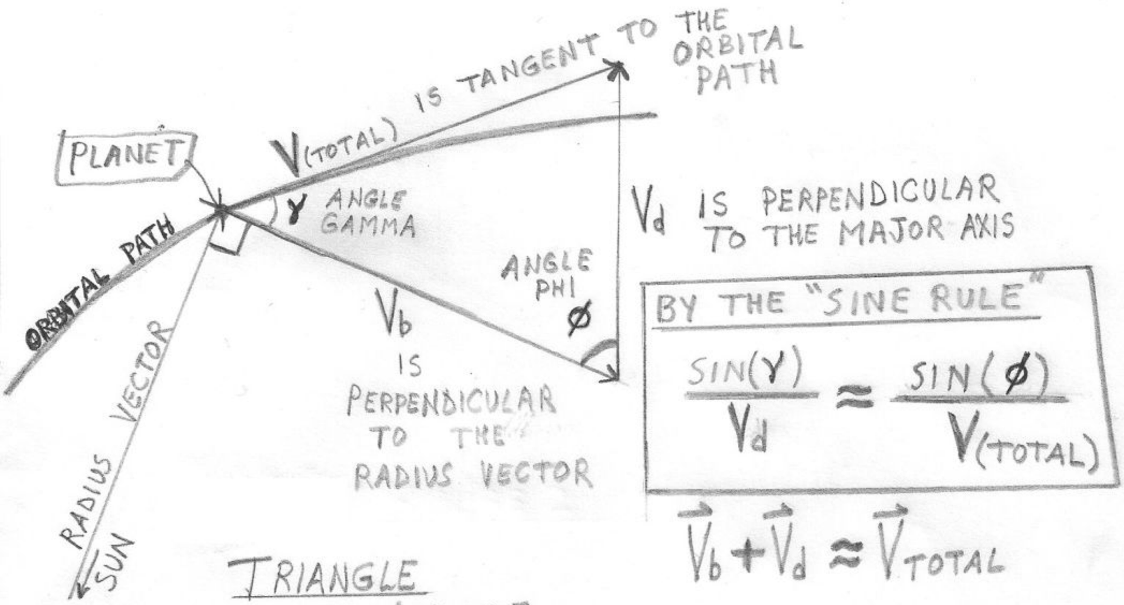
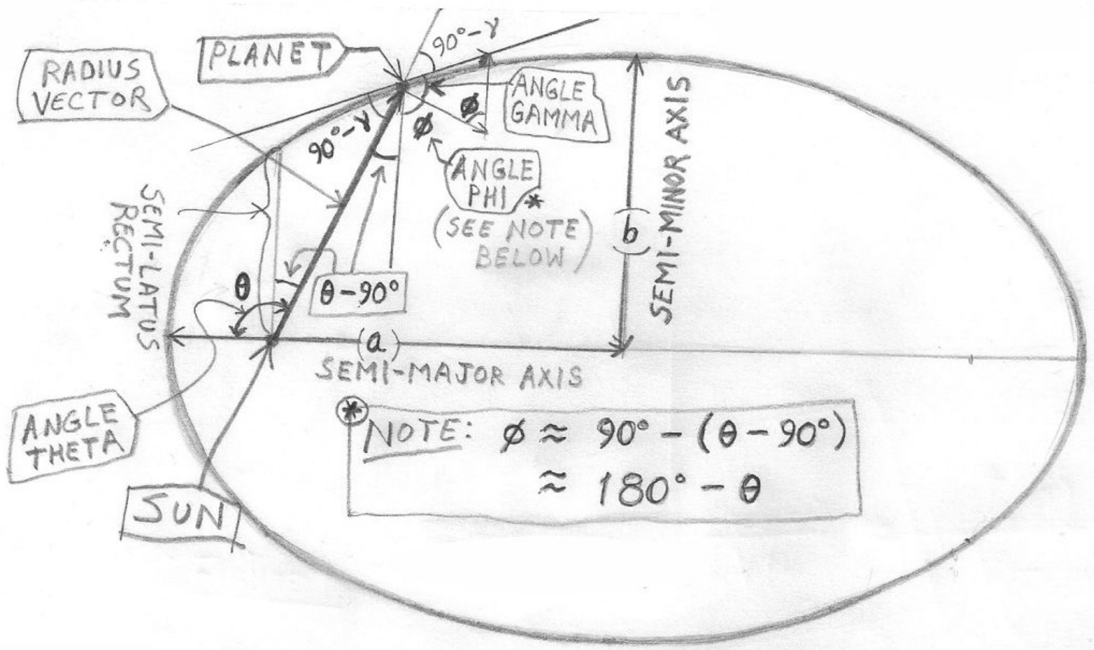


TRIANGLE COMPOSED OF THREE VELOCITY VECTORS

NOTE THAT V_{TOTAL} $\approx$ INSTANTANEOUS VELOCITY TANGENT TO THE ORBITAL PATH IS THE VECTOR SUM OF V_b AND V_d;

FIGURE 1



SCHEMATIC DIAGRAM OF AN ELLIPTICAL ORBIT

$\frac{h}{r} \approx 1 + e.\cos\theta$ THIS EQN. TOTALLY DESCRIBES AN ELLIPTICAL ORBIT

IN THIS PARTICULAR ORBIT,

$a \approx$ SEMI-MAJOR AXIS $\approx$ 11 MILLION KM

$b \approx$ SEMI-MINOR AXIS $\approx$ 7 MILLION KM

$\Rightarrow e \approx$ ECCENTRICITY = 0.771

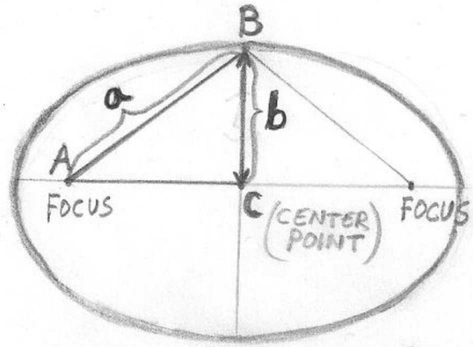
$\Rightarrow h \approx$ SEMI-LATUS RECTUM $\approx$ 4.455 M. KM

$\rightarrow r_p \approx$ PERIGEE RADIUS $\approx$ 2.515 M. KM

$\rightarrow r_a \approx$ APOGEE RADIUS $\approx$ 19.485 M. KM

FIGURE 2

FIGURE 3



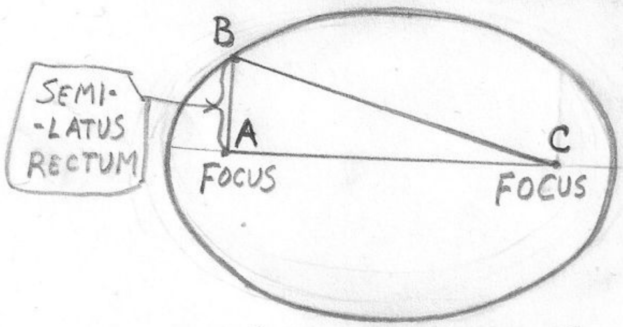
$$(AB)^2 \approx (AC)^2 + (BC)^2$$

$$\Rightarrow a^2 \approx (AC)^2 + b^2$$

$$\Rightarrow (AC) \approx (a^2 - b^2)^{1/2}$$

DERIVING

$h \approx b^2/a$



$$(BC)^2 \approx (AB)^2 + (AC)^2$$

$$[2a - h]^2 \approx h^2 + [2(a^2 - b^2)^{1/2}]^2$$

$$\Rightarrow h \approx b^2/a$$

NOTE THAT, IN BOTH ELLIPSES, THE SUM OF THE TWO LINES BETWEEN THE FOCUSUS AND THE POINT B IS EQUAL TO THE LENGTH OF THE MAJOR AXIS [2a];

FIGURE 3

essay: ELLIPTICAL ORBITS

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SUMMARY

In this essay I show how one can use easy maths to analyse planetary orbits, which are elliptical.
I also present an example of how one can use “the NASA method” to calculate planetary orbits, based on information in an interesting NASA document.

Key words: angular momentum, centrifugal, gravitational, Kepler, NASA, orbit, planet, planetary orbit, vis-viva equation;

INTRODUCTION

Recently I participated in a very interesting and spirited discussion, by email, regarding planetary orbits. I learned that these are not nearly as simple
as I had thought, and that there are several common errors which the average scientist might make when dealing with them. Sadly, these errors are
due to the fact that some textbooks and internet-sites give incorrect information re planetary orbits. The purpose of this essay is to possibly save others the
aggravation of making a common error in this context, and/or of having long and annoying arguments re this.

Part 1: CONSERVATION OF ANGULAR MOMENTUM

When two objects orbit around each other, the orbital angular momentum of each is constant, and does not change unless there is an input of energy
from outside the orbiting system, such as if an asteroid collides with a moon or planet.

This is easy to prove using Kepler's 2nd law, which says
that the radius-vector of an orbiting object sweeps through equal areas during equal times. One visualizes a very narrow triangle formed by two radius-vectors
separated by a small time ***dt.*** The area of such a triangle is ***A*** **= 1/2*r.r.w.dt,*** where "*r*" is radius and "*w*" is angular velocity. Note that the expression "*r.w.dt*"
represents the short distance which the planet moves during the time "*dt*."
If "*A*" and "*dt*" are constant, (which is another way to say "equal areas during equal times"), then ***r.r.w = r.v*** must also be constant.
{In German, "also" means "therefore"}
Because angular momentum is defined as ***ang.mom. = m.v.r = m.r.r.w,*** this means that angular momentum is constant.

Part 2: DO CENTRIFUGAL AND GRAVITATIONAL FORCES BALANCE EXACTLY ??

There is much confusion re this question, because some textbooks and internet-sites give incorrect explanations. However, using only some very simple maths, one can demonstrate that the two forces exactly balance at only two points in the orbit, unless the orbit is perfectly circular.

A basic equation which defines an elliptical orbit is:
h/r = 1 + e.cos(theta) (Eqn. 2.1), where "h" is the length of the semi-latus rectum, a constant; "r" is the length of the radius vector
between sun and planet, which varies unless the orbit is perfectly circular; "e" is the eccentricity of the orbit, a constant;
and "theta" is the angle between the radius vector and the major axis of the orbit, a variable **[Ref.#1].**

Eqn. 2.1 implies that: ***cos(theta) = (h/r - 1) / e*** **(Eqn. 2.1a);**
It also implies that: ***r = h / [1 + e.cos(theta)]*** **(Eqn. 2.1b),**
and ***r(perogee) = h / (1 + e)*** **(Eqn. 2.1c);**

Plus, if one differentiates both sides of Eqn. 2.1 with respect to time, one obtains:
- (h/r.r).dr/dt = - e.sin(theta).w, where "w" is angular velocity [d(theta)/dt] and "t" is time.
Note that one can drop the two minus-signs, and move "h/r.r" to the right side, to give:

dr/dt = [(r.r.e/h).sin(theta)].w **(Eqn. 2.2);**

Plus, Eqn. 2.2 implies that: ***w = angular velocity = [h / r.r.e.*****sin*****(theta)].dr/dt*** **(Eqn. 2.2a).**

Equation 2.3 is just simply the definition of the angular momentum of an orbiting planet:
L = m.[r.w].r **Eqn. 2.3,** where "L" is angular momentum. Note that "***r.w***" is the transverse velocity, (which is perpendicular to the radius vector), as distinct from the total [i.e., "instantaneous"] velocity (which is tangential to the orbit).
If the orbit is nearly circular, then these two velocities are nearly equal. In a more eccentric orbit, such as that of a comet,
there might be a considerable difference between transverse velocity and total velocity.

Eqn. 2.3 implies that **r.r. = L / m.w (Eqn. 2.3a);**

Combining Eqns. 2.2 and 2.3a gives: **dr/dt = [L / h.m.w].[e.sin(theta).w] = (L.e/m.h).sin(theta),**
which implies: **dr/dt = [(L.e)/(m.h)].sin(theta) (Eqn. 2.4);**

Differentiating again with respect to time gives a second-derivative expression:
d2r/dt2 = [(L.e)/(m.h)].cos(theta).w (Eqn. 2.4a);

Combining Eqns. 2.4a + 2.1a, and simplifying, one obtains:
m.(d2r/dt2) = [L/r].w - [L/h].w (Eqn. 2.5);

Multiplying both numerator and denominator on the right side by "L" gives:
m.(d2r/dt2) = [L.L/(L.r)].w - [L.L/(L.h)].w (Eqn. 2.5a);

Combining Eqns. 2.3 and 2.d5a, and simplifying: **m.(d2r/dt2) = L.L/(m.r.r.r) - L.L/(m.r.r.h) (Eqn. 2.6);**

The expression on the left side represents the planet's mass multiplied by an acceleration, the acceleration given by the 2nd derivative

of the radial distance, i.e., the radius, between the planet and its sun. It's the net force between the two orbiting objects.
One can visualize it as causing the length of the radius to increase and decrease as the planet moves along its non-circular orbit.
On the right side are expressions for centrifugal force and gravitational force respectively;
the gravitational force has a minus sign because it's attractive; i.e., because it tends to decrease the radius.

Inspection of Eqn.6 reveals that the net force is zero only when the two forces are equal, which happens only when **r = h,** i.e., at the two semi-latus rectum points.

This analysis shows that the gravitational force and the centrifugal force exactly balance each other at only two points
on an elliptical orbit, despite the fact that some textbooks say that they always exactly balance, as if the orbits were perfect circles,
which they are not. Worse yet, some textbooks say that centrifugal force is not a real force.

Note that this result is somewhat counter-intuitive:
until I looked at the maths, I was arguing that the two forces balance at perigee and apogee.

Part 3: THE VIS-VIVA EQUATION

This equation gives the total [i.e., "instantaneous"] velocity of the smaller body as a function of the mass of the larger body and
the semi-major axis of the orbit, both constants, and the radius, which varies as the planet goes around its orbit.
Again, one can use easy maths to derive it.

Because of the conservation of energy throughout the orbit, one can say that the total energy of the orbiting system is always equal to the sum of
the kinetic energy and the potential energy, using the convention that the potential energy is always negative, and therefore has a minus-sign in front of it:
V.V / 2 - GM / r = v(perigee).v(perigee) / 2 - GM / r(perigee) = v(apogee).v(apogee) / 2 - GM / r(apogee), **(Eqn. 3.1).**
Note that all the velocities are total velocities, tangential to the orbit.

One can rearrange this to say that
v(apogee).v(apogee) / 2 - v(perigee).v(perigee) / 2 = GM / r(apogee) - GM / r(perigee) **(Eqn. 3.2),**
where "*G*" is Newton's gravitational constant and "*M*" is the mass of the larger object.

Because total velocity and radius are perpendicular at perigee and apogee, the conservation of angular momentum requires that
v(perigee).r(perigee) = v(apogee).r(apogee) = constant.

So ***v(perigee) = v(apogee).[r(apogee)/r(perigee)]*** ***(*Eqn. 3.3).**

Combining eqns. 3.2 and 3.3 gives:
(1/2)*.v(apogee).v(apogee).*[1 - *r(apogee).r(apogee) / r(perigee).r(perigee)*] *= GM / r(apogee) - GM / r(perigee) (Eqn. 3.4);*
This implies:

{(1/2).[*r(perigee).r(perigee) - r(apogee).r(apogee)] / r(perigee).r(perigee)*}.*v(apogee).v(apogee) = GM / r(apogee) - GM / r(perigee),*
which implies:
(1/2).*v(apogee).v(apogee) = GM.*{ [*r(perigee) - r(apogee)] / r(perigee).r(apogee) }.{r(perigee).r(perigee) / [r(perigee).r(perigee) - r(apogee).r(apogee)]}*
= GM.r(perigee) / {r(apogee).[(r(perigee) + r(apogee)]};

From the geometry of an ellipse: ***r(apogee) + r(perigee) = 2.a,*** where "*a*" is the semi-major axis.
So: ***v(apogee).v(apogee) / 2 = (GM/2a).r(perigee) / r(apogee) = (GM/2a).[2a - r(apogee)] / r(apogee) = (GM/r(apogee)) - (GM/2a) (Eqn. 3.4a);***

Combining Eqns. 3.1 and 3.4a gives: ***Etotal = V.V / 2 - GM / r = -GM/2a,*** which leads to:

V.V = (GM/r - GM/2a).2 = GM.(2/r - 1/a), which is the vis-viva equation, and enables one to calculate the instantaneous velocity of a planet
at any point on its orbit, if one knows the radius ***[r].***

Part 4: THE "NASA METHOD"

One of the peculiarly interesting characteristics of elliptical orbits is that one can express the instantaneous-velocity vector as
the sum of two vectors, each of whose length is constant. One of these ***[Vd]*** always points in the same direction,
perpendicular to the major axis of the ellipse, and in the direction in which the planet moves on its nearest approach to its sun.
The other velocity vector ***[Vb]*** is always perpendicular to the radius vector, so that it rotates in a circle as the planet does an orbit.
See the illustration on page 115, also called page III-6, of **Ref.2,** which is a **NASA** document.
See also **Figures 1 and 2,** above.

To calculate the lengths of these two components of the total velocity, one can use the fact that the two vectors align to the same
orientation at the perigee and apogee points, when the planet is nearest and farthest from its sun. Thus:
Vb + Vd = v(perigee), and ***Vb - Vd = v(apogee);*** Solving these two equations simultaneously gives:
Vp = [v(perigee) + v(apogee)] / 2 **(Eqn. 3.5a),** and ***Vd = [v(perigee) - v(apogee)] / 2*** **(Eqn. 3.5b)*;***

Part 4a: the vector-sum triangle used in the NASA method
Note that one can construct a triangle which is composed of three velocity vectors: the total velocity ***[Vtotal]*** and the two component vectors ***[Vb*** and ***Vd],***
described above. Because none of these is perpendicular to either of the others, one uses the "cosine rule" in the calculation:
Vtotal = the square root of [Vb.Vb + Vd.Vd - 2.Vb.Vd.cos(phi)] (**Eqn. 3.6),**
where "*phi*" is the angle opposite the vector "***Vtotal***" in the triangle. Note that "***Vtotal***" is tangential to the orbit, "***Vb***" is perpendicular
to the radius vector, and "***Vd***" is perpendicular to the major axis, as illustrated in **Figures 1 and 2,** above, and in the **NASA** document **[Ref.2].**

Careful analysis of this illustration reveals that this angle "*phi*" is equal to 180 degrees minus the angle "*theta*" ---
which is the angle between the radius vector and the major axis, as described in Part 2, above. This means that **sin*(phi)* = sin*(theta)* (Eqn. 3.7a),**
and **cos*(phi)* = *-* cos*(theta)* (Eqn. 3.7b)*.***

One can now combine Eqns. 2.1a and 3.7b, to obtain: **cos*(theta)* = *(h/r - 1) / e* = *-* cos*(phi);*** so **cos*(phi)* = *(1 - h/r) / e* (Eqn. 3.8).**
One can now use the "cosine rule" (Eqn. 3.5) to calculate ***Vtotal:*** I.e., combining Eqns. 3.5 and 3.7 gives:
***Vtotal* = *the square root of [Vb.Vb + Vd.Vd - 2.Vb.Vd.[(1 - h/r) / e]* (Eqn. 3.8a),**
where ***h* = *b.b/a*** ("*b*" is semi-minor axis, "*a*" is semi-major axis), and ***e*** is defined as **the square root of [1 - b.b/a.a];**
{Please see **Appendix 1** for a derivation of the above expression for "*h*"}

Referring to **Figure 2,** one can see that ***dr/dt,*** the rate of change of the length of the radius, is equal to ***Vtotal.[*cos*(90 - gamma)]* = *Vtotal.[*sin*(gamma)]* (Eqn. 3.9)*,***
where ***gamma*** is the angle in the triangle which is opposite the velocity vector ***Vd.*** Note that the illustration on page 115, also called page III-6, in the **NASA** document,
also refers to this angle as ***gamma.***

One can use the "sine rule" to calculate the sine of this angle ***gamma***
as: **sin*(gamma)* = sin*(phi).[Vd/Vtotal]* (Eqn. 3.9a).** Note that **sin*(gamma)* = cos*(90 - gamma),*** so one has:
cos(90 - gamma)* = sin*(gamma)* = sin*(phi).[Vd/Vtotal]* (Eqn. 3.9b)*;
So one can say that ***dr/dt* = *Vtotal.*sin*(phi).[Vd/Vtotal]* = *Vd.*sin*(phi)* (Eqn. 3.9c)*.***
Using Eqn. 3.6a and the trigonometric identity **sin*(any angle)* = *square root of [1 -* cos*(any angle).*cos*(any angle)],*** one obtains:
sin*(phi)* = sin*(theta)* = *square root of {1 - [*cos*(theta)]^2};* Given that **cos*(theta)* = *(h/r - 1) / e* [Eqn. 2.1a]*,*** one obtains:
sin*(phi)* = *square root of {1 - [(h/r - 1) / e]^2},* which leads to, from Eqn. 3.9a:

sin*(gamma)* = *[square root of {1 - [(h/r - 1) / e]^2}].[Vd/Vtotal];* combining this with Eqn. 3.9, one sees that:

dr/dt* = *V(total).[square root of {1 - [(h/r - 1) / e]^2}].[Vd/Vtotal]}* = *Vd.[square root of {1 - [(h/r - 1) / e]^2}]* (Eqn. 3.10)*.

So: if one wants to know how quickly the length of the radius vector is changing at any point in the orbit, then one can use Eqn. 3.10; note that
one can use ***h = b.b/a*** [see **Appendix 1**] and ***e = sq.rt.[1 - b.b/a.a]*** [by definition] to show that the numeric value of ***1 - [(h/r - 1) / e]^2*** is zero at perigee;
this means that, as one would expect, ***dr/dt = zero*** at perigee. Likewise at apogee.

Part 5: SHOWING THAT *v(transverse)* IS THE CORRECT VELOCITY TO CALCULATE ANGULAR MOMENTUM

One can extend these simple maths to show that transverse velocity is in fact the correct velocity to use when calculating angular momentum.

The transverse velocity ***[v]*** is equal to ***r.[d(theta)/dt] = r.w,*** the radius vector multiplied by its angular velocity, each of which varies as it moves along its orbit.
Note that the transverse velocity at most points in a non-circular orbit is not the same as the total velocity, because the two are equal only at perigee and apogee,
because ***v*** is perpendicular to the radius vector only at those two points.

One can calculate the system's angular momentum if one knows the velocity and radius at perigee. From Eqn. 2.1b, one can calculate the radius at perigee as:
r(perigee) = h / [(1+e.cos(0)] = h/(1+e). But what about v(perigee) ?? At this point in the analysis it becomes more convenient to use numerical, rather than analytical,
methods. I.e., one can sketch a particular typical planetary orbit, as in **Figure 1,** below, whose semi-major axis [a] is 11 million kilometers [11 x 10^9 m],
and whose semi-minor axis [b] is 7 million kilometers [7 x 10^9 m]. From a and b, and the definition of eccentricity [e = sq.rt.{1 - (b.b)/(a.a)}],
one can calculate that the eccentricity [e] of the orbit is 0.771; from h = b.b/a [see **Appendix 1**], its semi-latus rectum is 4.455 x 10^9 m,
its r(perigee) is 2.515 x 10^9 m, and its r(apogee) is 19.485 x 10^9 m. One can use Eqn. 2.1c to calculate the last two numeric values.

After sketching this particular elliptical orbit, (and calculating its "vital statistics"), one can now choose a random v(perigee) for it, such as 1000 km/sec = 10^6 m/sec.
Plus, because of conservation of angular momentum, one can say that r(perigee).v(perigee) = r(apogee).v(apogee); given the r(perigee) and r(apogee) calculated
above, and the randomly chosen v(perigee) = 1.0 x 10^6 m/sec, one can calculate that v(apogee) = 0.129 x 10^6 m/sec. Using Eqns. 3.5a and 3.5b, one can
calculate numeric values for Vb and Vd: Vb = 0.565 x 10^6 m/sec and Vd = 0.435 x 10^6 m/sec. These are the two components of total velocity, as described above,
and in the NASA document **[Ref.2].**

From Eqn. 3.8a, one has an expression for V(total):
Vtotal = the square root of [Vb.Vb + Vd.Vd - 2.Vb.Vd.[(1 - h/r) / e] **(Eqn. 3.8a),**
Using this, one should be able to calculate V(total) at perigee and verify that it does in fact equal v(perigee), which we chose to be 10^6 m/sec.
Inserting into Eqn. 3.8a r = r(perigee) = 2.515 x 10^9 m, Vb = 0.565 x 10^6 m/sec, Vd = 0.435 x 10^6 m/sec, h = 4.455 x 10^9 m, and e = 0.771,
one obtains V(total) = 1.0 x 10^6 m/sec, verifying that NASA's method works.

{[If one wants to know the mass of the larger object in the system, one can use the vis-viva equation [Part 3, above] to calculate the numeric value of
GM, from which one can calculate ***M***]}

Using v(perigee) = 1.0 x 10^6 m/sec, the system's angular momentum is
given by: L = (m).v(perigee).r(perigee) = (m).(10^6 m/sec).(2.515 x 10^9 m) = (m).(2.515 x 10^(15) m.m/sec).

Finally, to show that the transverse velocity is the correct velocity to calculate angular momentum, one can choose a random point on the orbit,
and calculate the transverse velocity there, which is given by ***v(transverse) = r.w***, where ***w*** is angular velocity. Using (m).v(transverse).(r) = angular momentum,
one can then calculate angular momentum in this alternative way, and compare the result with the angular momentum calculated above.

To do this, one needs to develop an equation for angular velocity ***w***. From Eqn. 2.2a, one has:
w = angular velocity = [h / r.r.e.**sin*****(theta)].dr/dt*** **(Eqn. 2.2a).**
And from Eqn. 3.10: ***dr/dt = Vd.{square root of {1 - [(h/r - 1) / e]^2}}*** **(Eqn. 3.10).**
And from Eqn. 2.1a: ***cos(theta) = (h/r - 1) / e*** **(Eqn. 2.1a);**
Putting that all together gives: ***w = [h / r.r.e.*****sin*****(theta)].Vd.{sq.rt. of {1 - [*****cos*****(theta)]^2}}.***
Because sin(any angle) = sq.rt.of {1 - cos(any angle)^2}, one can simplify the above to say:
w = angular velocity = (h.Vd) / (r.r.e);
Using this, one can say that ang.mom. = m.[r.w].r = m.h.Vd / e = m.(4.455 x 10^9 m).(0.435 x 10^6 m/sec) / (0.771) =
(m).(2.514 x 10^15 m.m/sec), which agrees with the numeric value for angular momentum calculated above.

CONCLUSION

One can use easy maths to show the truth regarding several common points of confusion re details of planetary orbits, which are elliptical. One can use simple geometry to demonstrate that angular momentum of a planetary orbit does not change as it moves along the orbit, and that two forces are at play, (gravitational and centrifugal), which operate along the same line, and are equal at only two points on the orbit, unless the orbit is perfectly circular.

Plus, one can use these same easy maths to study the methods which NASA uses to track real satellites, as an interesting NASA document shows.

Appendix 1: Derivation of h = b.b / a

Please refer to **Figure 3** (above) for this derivation.
This calculation uses the well known fact that, for an ellipse, one can draw two straight lines, from any point on the ellipse to each of the two focuses, and that the sum of two such lines, connecting the two focuses to any point on the ellipse, is equal to the major axis, i.e., to the sum of the two semi-major axes, ***2a.*** If one chooses to draw the two lines from a point which is even with the center-point of the ellipse, so that the two lines are each the same length, then each is equal to the length of a semi-major axis. If one now draws a semi-minor axis, one has a right triangle whose hypotenuse is equal to ***a,*** and whose one side is equal to ***b.*** So by simple geometry the other side is equal to ***sq.rt.(a.a - b.b).*** Note that this other side connects a focus and the center-point of the ellipse. So one can say that the distance between the two foci is equal to twice this length, namely ***2.{sq.rt.(a.a - b.b)}.***

One can now construct another triangle, whose base is the line between the two foci, described above, and whose second side connects the principle focus and a semi-latus rectum point, and whose third side, the hypotenuse, is a line which connects the other focus to the same semi-latus rectum point. One knows that the length of the sum of the hypotenuse and the short side is equal to the length of the

major axis, i.e., ***2a,*** as described above. One also knows that the length of the short line is equal to that of the semi-latus rectum, i.e., ***h.***
So one knows that the length of the hypotenuse is ***2a - h.*** So, again using simple geometry, one has:
(2a - h)^2 = h^2 + [2.{sq.rt.(a.a - b.b)]^2 ... Solving this for ***h*** gives ***h = (b.b / a).***

Please note that this, and many other relationships between the parameters which describe an ellipse, are given in Table 1 of **Ref.2,**
which starts on page 125, also called page III-16.

REFERENCES

(1) internet-site: **http://www.bogan.ca/orbits/kepler/orbteqtn.html**
Note that Eqn.1 appears in the 6th row of the 2nd table at the internet-site above, where they refer to the semi-latus rectum as "p";
(2) internet-site: **Orbital Flight Handbook, volume 1 (NASA SP 33 PART 1),**
https://ntrs.nasa.gov/archive/nasa/casi.ntrs.nasa.gov/19630011221.pdf
https://www.dropbox.com/s/q89v2ut7nfnmthb/Orbital%20Flight%20Handbook%20-%20NASA.pdf?dl=0

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