A basic evidence of both Catalan-Mihailescu and Fermat-Wiles theorems and generalization to Fermat-Catalan and Beal conjectures

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Abstract

(MSC=11D04) We begin with an equation : $y^p = x^q \pm z^c$ (GCD(x,y)=1; GCD(x,z)=1, GCD(y,z)=1) and solve it.

(Keywords : Diophantine equations, Fermat-Catalan equation, solution)

Introduction

The goal of this document is clearly to solve the Fermat-Catalan equation $y^p = x^q \pm z^c$ (GCD(x,y)=1; GCD(x,z)=1, GCD(y,z)=1), in fact, to prove that one of the exponents must be even for having a solution. We know some solutions

$1^m + 2^3 = 3^2$

$2^5 + 7^2 = 3^4$

$13^2 + 7^3 = 2^9$

$2^7 + 17^3 = 71^2$

$3^5 + 11^4 = 122^2$

$33^8 + 1549034^2 = 65^7$

$9262^3 + 15312283^2 = 113^7$

$17^7 + 76271^3 = 21063928^2$

$43^8 + 96222^3 = 30042907^2$

If we study minutiously those solutions, it appears a common point, there is an even exponent in the formulas. Effectively, this even exponent appears at least in two other Diophantine equations : the Fermat equation, of course, but also in the Catalan equation and in some Pillai equations of the form $y^p = x^q + a$.

Our goal, here, is to show and to prove formally, with the tools of the logic and algebra, how this even exponent appears in the equations !

Resolution of Fermat-Catalan equation

Let the Fermat-Catalan equation :

$y^p = x^q + az^c$, $a = \pm 1$

Now, let

$$w = \frac{\log(-az^c + \sqrt{z^{2c} + 4y^p x^q}) - \log(2y^{p-2k})}{\log(x)}$$

$$\Rightarrow w\log(x) = \log(x^w) = \log(-az^c + \sqrt{z^{2c} + 4y^p x^q}) - \log(2y^{p-2k})$$

$$= \log(\frac{-az^c + \sqrt{z^{2c} + 4y^p x^q}}{2y^{p-2k}})$$

$$\Rightarrow x^w = \frac{-az^c + \sqrt{z^{2c} + 4y^p x^q}}{2y^{p-2k}}$$

$$\Rightarrow 2x^w y^{p-2k} + az^c = \sqrt{z^{2c} + 4y^p x^q}$$

$$\Rightarrow (2x^w y^{p-2k} + az^c)^2 = z^{2c} + 4y^p x^q$$

$$= 4x^{2w} y^{2p-4k} + z^{2c} + 4ax^w y^{p-2k} z^c$$

$$\Rightarrow x^{2w} y^{2p-4k} + ax^w y^{p-2k} z^c = y^p x^q$$

$$\Rightarrow x^w y^{p-2k} + az^c = y^{2k} x^{q-w}$$

$$\Rightarrow y^{2k} x^{q-w} - x^w y^{p-2k} = az^c = y^p - x^q = y^{2k} y^{p-2k} - x^w x^{q-w}$$

$$\Rightarrow y^{2k}(x^{q-w} - y^{p-2k}) + x^w(x^{q-w} - y^{p-2k}) = 0$$

$$= (y^{2k} + x^w)(x^{q-w} - y^{p-2k}) = 0$$

$$\Rightarrow y^{p-2k} = x^{q-w}$$

And

$$w' = \frac{\log(y^p - x^q + \sqrt{(y^p - x^q)^2 + 4y^p x^q}) - \log(2x^{p-2k})}{\log(y)}$$

$$\Rightarrow w'\log(y) = \log(y^{w'}) = \log(y^p - x^q + \sqrt{(y^p - x^q)^2 + 4y^p x^q}) - \log(2x^{p-2k})$$

$$= \log(\frac{y^q - x^p + \sqrt{(y^q - x^p)^2 + 4y^p x^p}}{2x^{p-2k}})$$

$$\Rightarrow y^{w'} = \frac{y^q - x^p + \sqrt{(y^q - x^p)^2 + 4y^q x^p}}{2x^{p-2k}}$$

$$\Rightarrow (2y^{w'} x^{p-2k} - (y^q - x^p))^2 = (y^q - x^p)^2 + 4y^q x^p$$

$$= 4y^{2w'} x^{2p-4k} + (y^q - x^p)^2 - 4y^{w'} x^{p-2k}(y^q - x^p)$$

$$\Rightarrow y^{2w'}x^{2p-4k} - y^{w'}x^{p-2k}(y^q - x^p) = y^q x^p$$

$$\Rightarrow y^{w'}x^{p-2k} - (y^q - x^p) = x^{2k}y^{q-w'}$$
$$\Rightarrow x^{2k}y^{q-w'} - y^{w'}x^{p-2k} = -y^q + x^p = -y^{w'}y^{q-w'} + x^{2k}x^{p-2k}$$
$$\Rightarrow (x^{2k} + y^{w'})(x^{p-2k} - y^{q-w'}) = 0$$
$$\Rightarrow x^{p-2k} = y^{q-w'}$$

But

$$x^{(p-2k)^2} = y^{(p-2k)(q-w')} = x^{(q-w)(q-w')}$$
$$\Rightarrow (p-2k)^2 = (q-w)(q-w')$$
$$\Rightarrow x^{q-w} = y^{p-2} = y^{\sqrt{(q-w)(q-w')}}$$
$$\Rightarrow x^{\sqrt{q-w}} = y^{\sqrt{q-w'}}$$

And

$$w_1 = q - \frac{(q-w)^2}{p-2k}$$
$$w_2 = q - \frac{(q-w')^2}{p-2k}$$
$$(q-w)^2 = (p-2k)(q-w_1)$$
$$(q-w')^2(p-2k)(q-w_2)$$
$$(q-w)^2(q-w')^2 = (p-2k)^4 = (p-2k)^2(q-w_1)(q-w_2)$$

$$\Rightarrow (q-w_1)(q-w_2) = (p-2k)^2 = (q-w)(q-w')$$

If $(p-2k)(q-w)(q-w') \neq 0$

$$\Rightarrow (q-w)^2 = (p-2k)(q-w_1) = \sqrt{(q-w_1)^3(q-w_2)} = \sqrt{(q-w)(q-w')}(q-w_1)$$
$$\Rightarrow q - w_1 = (q-w)\sqrt{\frac{q-w}{q-w'}}$$

$$\Rightarrow q - w_2 = (q-w')\sqrt{\frac{q-w'}{q-w}}$$

$$y^{q-w_2} = y^{(q-w')\sqrt{\frac{q-w'}{q-w}}} = x^{(p-2k)\sqrt{\frac{q-w'}{q-w}}} = x^{q-w'} = x^{\sqrt{(p-2k)(q-w_2)}}$$
$$\Rightarrow y^{q-w_2} = x^{q-w'} = x^{\sqrt[4]{(q-w_1)(q-w_2)^3}}$$

$$\Rightarrow y^{\sqrt[4]{q-w_2}} = x^{\sqrt[4]{q-w_1}}$$
$$\Rightarrow x^{q-w_1} = y^{q-w}$$

But

$$-w_2 + w' = -(q-w') + (q-w')\sqrt{\frac{q-w'}{q-w}} = (q-w')\frac{\sqrt{q-w'} - \sqrt{q-w}}{\sqrt{q-w}}$$

$$-w + w_1 = q - w - (q-w)\sqrt{\frac{q-w}{q-w'}} = (q-w)\frac{\sqrt{q-w'} - \sqrt{q-w}}{\sqrt{q-w'}}$$

$$\Rightarrow \frac{-w_2 + w'}{-w + w_1} = (\sqrt{\frac{q-w'}{q-w}})^3$$

$$w_2 - w = q - w - (q-w')\sqrt{\frac{q-w'}{q-w}} = \frac{(q-w)\sqrt{q-w} - (q-w')\sqrt{q-w'}}{\sqrt{q-w}}$$

$$w' - w_1 = -(q-w') + (q-w)\sqrt{\frac{q-w}{q-w'}} = \frac{(q-w)\sqrt{q-w} - (q-w')\sqrt{q-w'}}{\sqrt{q-w'}}$$

$$\Rightarrow \frac{w_2 - w}{w' - w_1} = \sqrt{\frac{q-w'}{q-w}}$$

$$\Rightarrow (-w_2 + w')(w' - w_1)^3 = (w_2 - w)^3(-w + w_1)$$

$$\Rightarrow w^4 - w'^4 + w' w_1^3 - w w_2^3 + 3w_1 w'^3 - 3w_2 w^3 - 3w_1^2 w'^2 + 3w_2^2 w^2 +$$
$$+ w_2 w'^3 - w_1 w^3 - w_2 w_1^3 + w_1 w_2^3 - 3w_2 w_1 w'^2 + 3w_2 w_1 w^2 + 3w' w_2 w_1^2 - 3w w_1 w_2^2 = 0$$

$$w' = kw$$

$$(1-k^4)w^4 + 3kw(w_1^3 - w_2^3) + 3w^3(k^3 w_1 - w_2) + 3w^2(w_2^2 - k^2 w_1^2) +$$
$$+ w^3(k^3 w_2 - w_1) + w_2 w_1(-w_1^2 + w_2^2) + 3w_2 w_1(k^2-1)w^2 + 3w w_2 w_1(k w_1 - w_2) = 0$$
$$= (1-k^4)w^4 + 3kw(w_1^3 - w_2^3) + 3w^3(k^3-1)w_1 + w_1 - w_2) + 3w^2(w_2^2 - w_1^2 + (1-k^2)w_1^2) +$$
$$+ w^3((k^3-1)w_2 + w_2 - w_1) + w_2 w_1(-w_1^2 + w_2^2) + 3w_2 w_1(k^2-1)w^2 + 3w w_2 w_1((k-1)w_1 + w_1 - w_2) = 0$$

$$w_2 = k' w_1$$

$$w_2 - w_1 = (k'-1)w_1 = \frac{(q-w')^2 - (q-w)^2}{p-2k} = \frac{(1-k)w(2q-w-w')}{p-2k}$$

$$\Rightarrow (1-k^4)w^4 + 3kw(w_1 - w_2)(w_1^2 + w_1 w_2 + w_2^2) + 3w^3((k^3-1)w_1 + w_1 - w_2) + 3w^2((w_2 - w_1)(w_2 + w_1) + (1-k^2)w_1^2) +$$
$$+ w^3((k^3-1)w_2 + w_2 - w_1) + w_2 w_1(w_1 + w_2)(w_2 - w_1) + 3w_2 w_1(k^2-1)w^2 + 3w w_2 w_1((k-1)w_1 + w_1 - w_2) = 0$$
$$= (1-k^4)w^4 - 3kw(k'-1)w_1(w_1^2 + w_1 w_2 + w_2^2) + 3w^3((k^3-1)w_1 - (k'-1)w_1) + 3w^2((k'-1)w_1(w_2 + w_1) + (1-k^2)w_1^2) +$$
$$+ w^3((k^3-1)w_2 + (k'-1)w_1) + w_2 w_1(w_1 + w_2)(k'-1)w_1 + 3w_2 w_1(k^2-1)w^2 + 3w w_2 w_1((k-1)w_1 - (k'-1)w_1) = 0$$
$$\Rightarrow (p-2k)(1-k^4)w^4 - 3kw(1-k)w(2q-w-w')(w_1^2 + w_1 w_2 + w_2^2) + 3w^3((k^3-1)w_1(p-2k) - (1-k)w(2q-w-w')) +$$
$$+ 3w^2((1-k)w(2q-w-w')(w_2 + w_1) + (p-2)(1-k^2)w_1^2) +$$
$$+ w^3((k^3-1)w_2(p-2k) + (1-k)w(2q-w-w')) + w_2 w_1(w_1 + w_2)(1-k)w(2q-w-w') +$$
$$+ 3w_2 w_1(k^2-1)w^2(p-2k) + 3w w_2 w_1((k-1)w_1(p-2k) - (1-k)w(2q-w-w')) = 0$$
$$= (1-k)((1+k+k^2+k^3)(p-2k)w^3 - (3kw(2q-w-w')w_1^2(k'^2 + k'+1)) + 3w^2((k^2+k+1)w_1(p-2k) - w(2q-w-w')) +$$
$$+ 3w((2q-w-w')w(k'+1)w_1 + (p-2k)(1+k)w_1^2) +$$
$$+ w^2((k^2+k+1)k' w_1(p-2k) + w(2q-w-w')) + k' w_1^3(k'+1)(2q-w-w') +$$
$$+ 3k' w_1^2(k+1)w(p-2k) + 3k' w_1^2(w_1(p-2k) - w(2q-w-w')) = 0$$

$$\Rightarrow k - 1 = 0 = k' - 1$$

$$\Rightarrow q-w=q-w' \Rightarrow (p-2k)^2=(q-w)(q-w')=(q-w)^2$$
$$\gcd(x,y)=1 \Rightarrow y^{p-2k}=x^{q-w}=x^{p-2} \Rightarrow q-w=q-w'=p-2k=0$$

Now, if there is an odd solution : if we pose p=2k'+1, with the same calculus, it leads to 2k=2k'+1 wich is impossible. The equation has solution for p even or for p odd exclusively. As we have yet solutions with p even, it means that p can not be odd !

$$\Rightarrow p=2k$$

If

$$k=2k'+1 \Rightarrow (y^2)^{(2k'+1)} \neq x^q+az^c \Rightarrow p=2^{k"}$$

For Fermat equation

$$x^n+y^n=z^n$$
$$\Rightarrow n=2k$$

$$(x^2)^{(2k'+1)}+(y^2)^{(2k'+1)} \neq (z^2)^{(2k'+1)}$$

$$k=2k"$$
$$(x^{k"})^4+(y^{k"})^4 \neq (z^{k"})^4 \Rightarrow n=2$$

For Catalan-Mikhailescu theorem, it has been proved that the only solution for p=2k is p=2.

Conclusion

Fermat-Catalan equation $y^p=x^q \pm z^c$ has solutions only for one of the exponents even with GCD(x,y)=GCD(x,z)=GCD(y,z)=1 and p>1, q>1, a>1.