

# A basic evidence of both Catalan-Mihailescu and Fermat-Wiles theorems and generalization to Fermat-Catalan and Beal conjectures

Jamel Ghanouchi

## Abstract

(MSC=11D04) We begin with an equation :  $y^p = x^q \pm z^c$  ( $\text{GCD}(x,y)=1$ ;  $\text{GCD}(x,z)=1$ ,  $\text{GCD}(y,z)=1$ ) and solve it.

(Keywords : Diophantine equations, Fermat-Catalan equation, solution)

## Introduction

The goal of this document is clearly to solve the Fermat-Catalan equation  $y^p = x^q \pm z^c$  ( $\text{GCD}(x,y)=1$ ;  $\text{GCD}(x,z)=1$ ,  $\text{GCD}(y,z)=1$ ), in fact, to prove that one of the exponents must be even for having a solution. We know some solutions

$$1^m + 2^3 = 3^2$$

$$2^5 + 7^2 = 3^4$$

$$13^2 + 7^3 = 2^9$$

$$2^7 + 17^3 = 71^2$$

$$3^5 + 11^4 = 122^2$$

$$33^8 + 1549034^2 = 65^7$$

$$9262^3 + 15312283^2 = 113^7$$

$$17^7 + 76271^3 = 21063928^2$$

$$43^8 + 96222^3 = 30042907^2$$

If we study minutiously those solutions, it appears a common point, there is an even exponent in the formulas. Effectively, this even exponent appears at least in two other Diophantine equations : the Fermat equation, of course, but also in the Catalan equation and in some Pillai equations of the form  $y^p = x^q + a$ .

Our goal, here, is to show and to prove formally, with the tools of the logic and algebra, how this even exponent appears in the equations !

## Resolution of Fermat-Catalan equation

Let the Fermat-Catalan equation :

$$y^p = x^q + az^c, \quad a = \pm 1$$

Now, let

$$\begin{aligned} w &= \frac{\log(-az^c + \sqrt{z^{2c} + 4y^p x^q}) - \log(2y^{p-2k})}{\log(x)} \\ \Rightarrow w \log(x) &= \log(x^w) = \log(-az^c + \sqrt{z^{2c} + 4y^p x^q}) - \log(2y^{p-2k}) \\ &= \log\left(\frac{-az^c + \sqrt{z^{2c} + 4y^p x^q}}{2y^{p-2k}}\right) \\ \Rightarrow x^w &= \frac{-az^c + \sqrt{z^{2c} + 4y^p x^q}}{2y^{p-2k}} \\ \Rightarrow 2x^w y^{p-2k} + az^c &= \sqrt{z^{2c} + 4y^p x^q} \\ \Rightarrow (2x^w y^{p-2k} + az^c)^2 &= z^{2c} + 4y^p x^q \\ &= 4x^{2w} y^{2p-4k} + z^{2c} + 4ax^w y^{p-2k} z^c \\ \Rightarrow x^{2w} y^{2p-4k} + ax^w y^{p-2k} z^c &= y^p x^q \\ \Rightarrow x^w y^{p-2k} + az^c &= y^{2k} x^{q-w} \\ \Rightarrow y^{2k} x^{q-w} - x^w y^{p-2k} &= az^c = y^p - x^q = y^{2k} y^{p-2k} - x^w x^{q-w} \\ \Rightarrow y^{2k} (x^{q-w} - y^{p-2k}) + x^w (x^{q-w} - y^{p-2k}) &= 0 \\ &= (y^{2k} + x^w)(x^{q-w} - y^{p-2k}) = 0 \\ \Rightarrow y^{p-2k} &= x^{q-w} \end{aligned}$$

And

$$\begin{aligned} w' &= \frac{\log(y^p - x^q + \sqrt{(y^p - x^q)^2 + 4y^p x^q}) - \log(2x^{p-2k})}{\log(y)} \\ \Rightarrow w' \log(y) &= \log(y^{w'}) = \log(y^p - x^q + \sqrt{(y^p - x^q)^2 + 4y^p x^q}) - \log(2x^{p-2k}) \\ &= \log\left(\frac{y^q - x^p + \sqrt{(y^q - x^p)^2 + 4y^q x^p}}{2x^{p-2k}}\right) \\ \Rightarrow y^{w'} &= \frac{y^q - x^p + \sqrt{(y^q - x^p)^2 + 4y^q x^p}}{2x^{p-2k}} \\ \Rightarrow (2y^{w'} x^{p-2k} - (y^q - x^p))^2 &= (y^q - x^p)^2 + 4y^q x^p \\ &= 4y^{2w'} x^{2p-4k} + (y^q - x^p)^2 - 4y^{w'} x^{p-2k} (y^q - x^p) \end{aligned}$$

$$\begin{aligned}
&\Rightarrow y^{2w'} x^{2p-4k} - y^{w'} x^{p-2k} (y^q - x^p) = y^q x^p \\
&\Rightarrow y^{w'} x^{p-2k} - (y^q - x^p) = x^{2k} y^{q-w'} \\
&\Rightarrow x^{2k} y^{q-w'} - y^{w'} x^{p-2k} = -y^q + x^p = -y^{w'} y^{q-w'} + x^{2k} x^{p-2k} \\
&\Rightarrow (x^{2k} + y^{w'})(x^{p-2k} - y^{q-w'}) = 0 \\
&\Rightarrow x^{p-2k} = y^{q-w'}
\end{aligned}$$

But

$$\begin{aligned}
x^{(p-2k)^2} &= y^{(p-2k)(q-w')} = x^{(q-w')(q-w')} \\
&\Rightarrow (p-2k)^2 = (q-w')(q-w') \\
&\Rightarrow x^{q-w'} = y^{p-2} = y^{\sqrt{(q-w')(q-w')}} \\
&\Rightarrow x^{\sqrt{q-w'}} = y^{\sqrt{q-w'}}
\end{aligned}$$

And

$$\begin{aligned}
w_1 &= q - \frac{(q-w)^2}{p-2k} \\
w_2 &= q - \frac{(q-w')^2}{p-2k} \\
(q-w)^2 &= (p-2k)(q-w_1) \\
(q-w')^2 &= (p-2k)(q-w_2) \\
(q-w)^2 (q-w')^2 &= (p-2k)^4 = (p-2k)^2 (q-w_1)(q-w_2) \\
&\Rightarrow (q-w_1)(q-w_2) = (p-2k)^2 = (q-w)(q-w')
\end{aligned}$$

if  $(p-2k)(q-w)(q-w') \neq 0$

$$\begin{aligned}
&\Rightarrow (q-w)^2 = (p-2k)(q-w_1) = \sqrt{(q-w_1)^3 (q-w_2)} = \sqrt{(q-w)(q-w')}(q-w_1) \\
&\Rightarrow q-w_1 = (q-w) \sqrt{\frac{q-w}{q-w'}} \\
&\Rightarrow q-w_2 = (q-w') \sqrt{\frac{q-w'}{q-w}} \\
y^{q-w_2} &= y^{(q-w') \sqrt{\frac{q-w'}{q-w}}} = x^{(p-2k) \sqrt{\frac{q-w'}{q-w}}} = x^{q-w'} = x^{\sqrt{(p-2k)(q-w_2)}} \\
&\Rightarrow y^{q-w_2} = x^{q-w'} = x^{\sqrt[4]{(q-w_1)(q-w_2)^3}} \\
&\Rightarrow y^{\sqrt[4]{q-w_2}} = x^{\sqrt[4]{q-w_1}} \\
&\Rightarrow x^{q-w_1} = y^{q-w}
\end{aligned}$$

But

$$-w_2 + w' = -(q - w') + (q - w') \sqrt{\frac{q - w'}{q - w}} = (q - w') \frac{\sqrt{q - w'} - \sqrt{q - w}}{\sqrt{q - w}}$$

$$-w + w_1 = q - w - (q - w) \sqrt{\frac{q - w}{q - w'}} = (q - w) \frac{\sqrt{q - w'} - \sqrt{q - w}}{\sqrt{q - w'}}$$

$$\Rightarrow \frac{-w_2 + w'}{-w + w_1} = \left( \sqrt{\frac{q - w'}{q - w}} \right)^3$$

$$w_2 - w = q - w - (q - w') \sqrt{\frac{q - w'}{q - w}} = \frac{(q - w) \sqrt{q - w} - (q - w') \sqrt{q - w'}}{\sqrt{q - w}}$$

$$w' - w_1 = -(q - w') + (q - w) \sqrt{\frac{q - w}{q - w'}} = \frac{(q - w) \sqrt{q - w} - (q - w') \sqrt{q - w'}}{\sqrt{q - w'}}$$

$$\Rightarrow \frac{w_2 - w}{w' - w_1} = \sqrt{\frac{q - w'}{q - w}}$$

$$\Rightarrow (-w_2 + w')(w' - w_1)^3 = (w_2 - w)^3(-w + w_1)$$

$$\begin{aligned} \Rightarrow w^4 - w'^4 + w' w_1^3 - w w_2^3 + 3w_1 w'^3 - 3w_2 w^3 - 3w_1^2 w'^2 + 3w_2^2 w^2 + \\ + w_2 w'^3 - w_1 w^3 - w_2 w_1^3 + w_1 w_2^3 - 3w_2 w_1 w'^2 + 3w_2 w_1 w^2 + 3w' w_2 w_1^2 - 3w w_1 w_2^2 = 0 \\ w' = kw \end{aligned}$$

$$\begin{aligned} (1 - k^4)w^4 + 3kw(w_1^3 - w_2^3) + 3w^3(k^3 w_1 - w_2) + 3w^2(w_2^2 - k^2 w_1^2) + \\ + w^3(k^3 w_2 - w_1) + w_2 w_1(-w_1^2 + w_2^2) + 3w_2 w_1(k^2 - 1)w^2 + 3w w_2 w_1(k w_1 - w_2) = 0 \\ = (1 - k^4)w^4 + 3kw(w_1^3 - w_2^3) + 3w^3(k^3 - 1)w_1 + w_1 - w_2) + 3w^2(w_2^2 - w_1^2 + (1 - k^2)w_1^2) + \\ + w^3((k^3 - 1)w_2 + w_2 - w_1) + w_2 w_1(-w_1^2 + w_2^2) + 3w_2 w_1(k^2 - 1)w^2 + 3w w_2 w_1((k - 1)w_1 + w_1 - w_2) = 0 \end{aligned}$$

$$w_2 = k' w_1$$

$$w_2 - w_1 = (k' - 1)w_1 = \frac{(q - w')^2 - (q - w)^2}{p - 2k} = \frac{(1 - k)w(2q - w - w')}{p - 2k}$$

$$\begin{aligned} \Rightarrow (1 - k^4)w^4 + 3kw(w_1 - w_2)(w_1^2 + w_1 w_2 + w_2^2) + 3w^3((k^3 - 1)w_1 + w_1 - w_2) + 3w^2((w_2 - w_1)(w_2 + w_1) + (1 - k^2)w_1^2) + \\ + w^3((k^3 - 1)w_2 + w_2 - w_1) + w_2 w_1(w_1 + w_2)(w_2 - w_1) + 3w_2 w_1(k^2 - 1)w^2 + 3w w_2 w_1((k - 1)w_1 + w_1 - w_2) = 0 \\ = (1 - k^4)w^4 - 3kw(k' - 1)w_1(w_1^2 + w_1 w_2 + w_2^2) + 3w^3((k^3 - 1)w_1 - (k' - 1)w_1) + 3w^2((k' - 1)w_1(w_2 + w_1) + (1 - k^2)w_1^2) + \\ + w^3((k^3 - 1)w_2 + (k' - 1)w_1) + w_2 w_1(w_1 + w_2)(k' - 1)w_1 + 3w_2 w_1(k^2 - 1)w^2 + 3w w_2 w_1((k - 1)w_1 - (k' - 1)w_1) = 0 \\ \Rightarrow (p - 2k)(1 - k^4)w^4 - 3kw(1 - k)w(2q - w - w')(w_1^2 + w_1 w_2 + w_2^2) + 3w^3((k^3 - 1)w_1(p - 2k) - (1 - k)w(2q - w - w')) + \\ + 3w^2((1 - k)w(2q - w - w')(w_2 + w_1) + (p - 2)(1 - k^2)w_1^2) + \\ + w^3((k^3 - 1)w_2(p - 2k) + (1 - k)w(2q - w - w')) + w_2 w_1(w_1 + w_2)(1 - k)w(2q - w - w') + \\ + 3w_2 w_1(k^2 - 1)w^2(p - 2k) + 3w w_2 w_1((k - 1)w_1(p - 2k) - (1 - k)w(2q - w - w')) = 0 \\ = (1 - k)((1 + k + k^2 + k^3)(p - 2k)w^3 - (3kw(2q - w - w')w_1^2(k^2 + k' + 1)) + 3w^2((k^2 + k + 1)w_1(p - 2k) - w(2q - w - w')) + \\ + 3w((2q - w - w')w(k' + 1)w_1 + (p - 2k)(1 + k)w_1^2) + \\ + w^2((k^2 + k + 1)k' w_1(p - 2k) + w(2q - w - w')) + k' w_1^3(k' + 1)(2q - w - w') + \\ + 3k' w_1^2(k + 1)w(p - 2k) + 3k' w_1^2(w_1(p - 2k) - w(2q - w - w')) = 0 \end{aligned}$$

$$\Rightarrow k - 1 = 0 = k' - 1$$

$$\Rightarrow q - w = q - w' \Rightarrow (p - 2k)^2 = (q - w)(q - w') = (q - w)^2$$

$$\gcd(x, y) = 1 \Rightarrow y^{p-2k} = x^{q-w} = x^{p-2} \Rightarrow q - w = q - w' = p - 2k = 0$$

Now, if there is an odd solution : if we pose  $p=2k'+1$ , with the same calculus, it leads to  $2k=2k'+1$  which is impossible. The equation has solution for  $p$  even or for  $p$  odd exclusively. As we have yet solutions with  $p$  even, it means that  $p$  can not be odd !

$$\Rightarrow p = 2k$$

If

$$k = 2k' + 1 \Rightarrow (y^2)^{(2k'+1)} \neq x^q + az^c \Rightarrow p = 2k''$$

For Fermat equation

$$x^n + y^n = z^n$$

$$\Rightarrow n = 2k$$

$$(x^2)^{(2k'+1)} + (y^2)^{(2k'+1)} \neq (z^2)^{(2k'+1)}$$

$$k = 2k''$$

$$(x^{k''})^4 + (y^{k''})^4 \neq (z^{k''})^4 \Rightarrow n = 2$$

For Catalan-Mikhailescu theorem, it has been proved that the only solution for  $p=2k$  is  $p=2$ .

## Conclusion

Fermat-Catalan equation  $y^p = x^q \pm z^c$  has solutions only for one of the exponents even with  $\gcd(x,y)=\gcd(x,z)=\gcd(y,z)=1$  and  $p>1, q>1, a>1$ .