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The Earth's Gravitational field

Abstract

The complete mathematical expression for the Earth's gravitationl field and potential is computed and examined.

Educational paper, requires basic abilities with differential calculus.

I always found this part lacking in the textbooks so I made my own lecture.

1) The Newton's Law

The law of gravitational interaction between masses is so widespreadly known that it feels even a little bit awkward to write it down for the n^{th} time. Anyway, here it is [1]

$$F = \frac{GM_1M_2}{x^2}$$

where G is a constant, M_1 and M_2 the masses and x the distance between the centers of the two bodies.
From this, one can define the Gravitational Field of a single body

$$E = \frac{GM_1}{x^2}$$

where x is the distance from the center of the body. Or not ???

Let's start with something very familiar, Earth.
The above formula holds and gives pretty good values for any practical purpose.
All bodies fall towards the center of Earth.
On the surface …
And in outer space too …
And here's where the catch is.

2) Drilling the hole

But, what happens in the inside?
How’s gravity below the surface?
We cannot do it for real, unluckily, but let’s assume we can and let’s drill a 6300 km long hole from the surface straight to the center of the planet.

How’s gravity there?
Does Newton’s formula hold?
No, it doesn’t.
Also the simmetry of the problem suggests the right answer. A body at the center of Earth wouldn’t know what direction to fall. There’s no gravity there. Nothing.

3) I don't believe it!

You might.
There's also a mathematical proof of that, just apply Gauss's theorem (the very same of electrodynamics, yes!). It'll show you the correct formula for the gravitational field INSIDE a massive body. A simple linear function.

$$E = ax + b$$

So this is were the subtlety was; you have two formulas, one for the external and one for the internal field

$$E_{internal} = ax + b$$
$$E_{external} = \frac{GM}{x^2}$$

But this is no big deal; we can also compute the unknown constant with a simple reasoning.
We assume the Force and the Field are null at the center, this gives b=0.
We also impose a continuity condition, at the surface

$$E_{internal}^{surface} = E_{external}^{surface}$$

Let's call R the Earth radius and M the Earth mass we have:

$$aR = \frac{GM}{R^2}$$

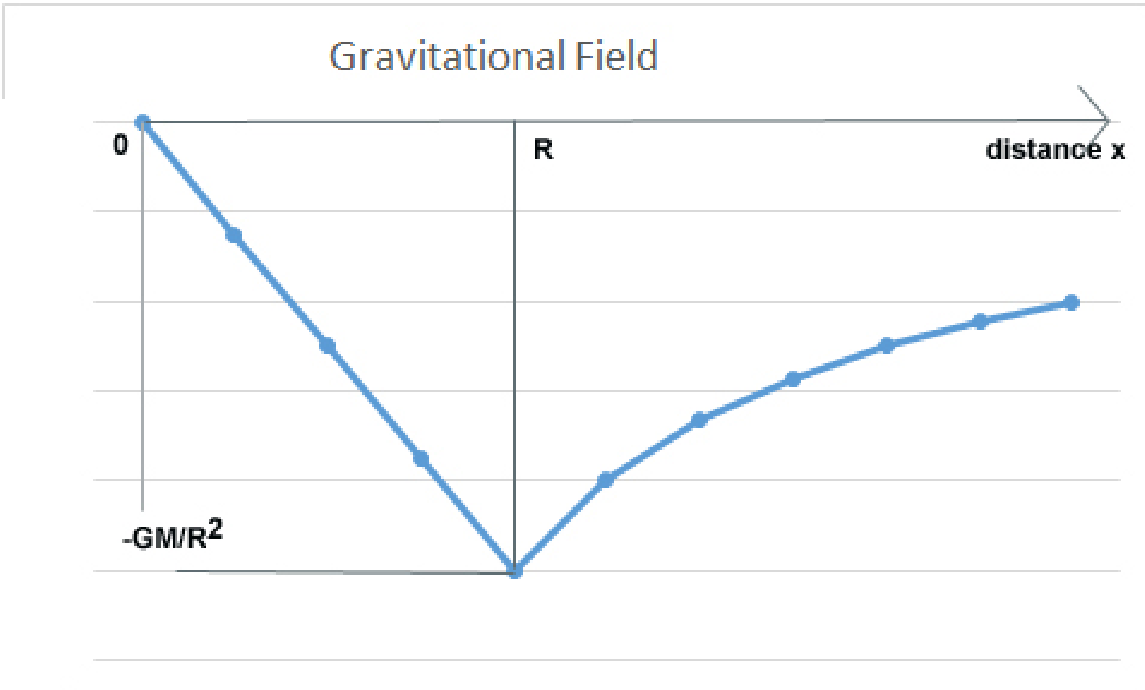
hence

$$a = \frac{GM}{R^3}$$

So these are our final formulas

$$E_{internal} = \frac{GM}{R^3}x$$
$$E_{external} = \frac{GM}{x^2}$$

The following graph gives the appropriate shape of the Earth's (or any other body) gravitational field; an appropriate (for an attractive field) minus sign is added.



Let's see also what happens to the potential, usually defined as

$$V(x) = -\int E(x)dx$$

$$V_{internal} = \frac{GM}{2R^3}x^2 + c1$$
$$V_{external=} -\frac{GM}{x} + c2$$

Again we can compute the constants with reasonable assumptions; c2=0 (V=0 to infinity) plus the usual continuity condition:

$$V_{internal}^{surface} = V_{external}^{surface}$$

This gives

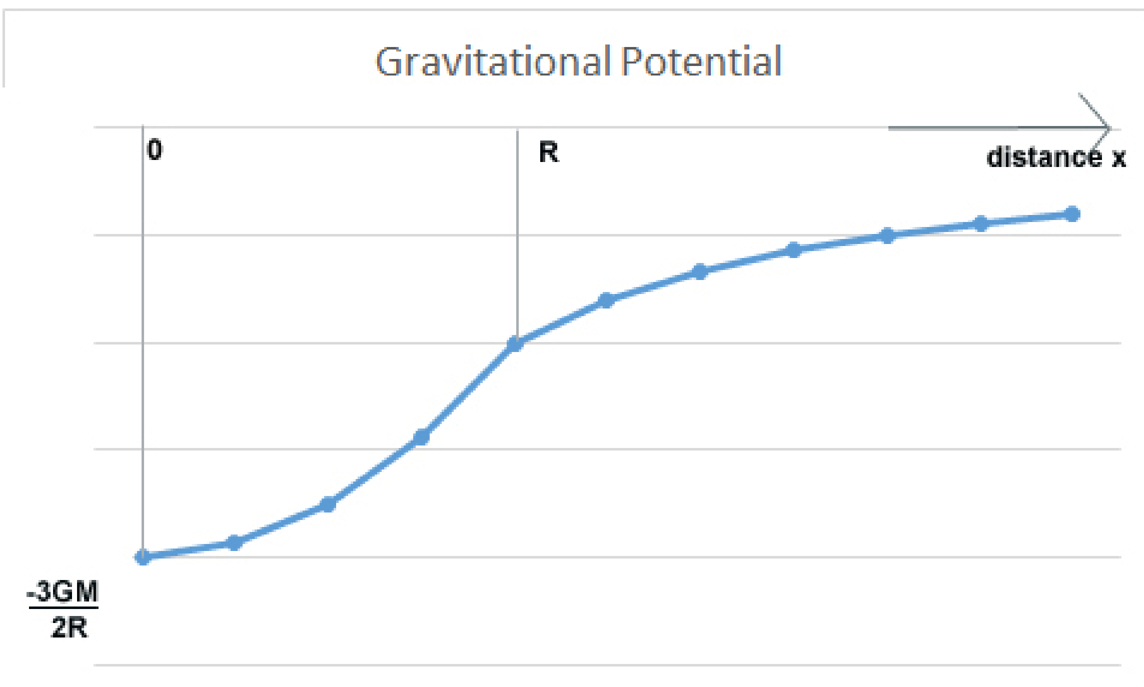
$$-\frac{GM}{2R} = \frac{GM}{R} + c1$$

So that

$$V_{internal} = \frac{GM}{2R^3}x^2 - \frac{3GM}{2R}$$

$$V_{external=} - \frac{GM}{x}$$

Depicted in the following graph



Let's have a look at the energy associated with the constant term; it can be divided in two parts as follows

$$U = Vm = \frac{3GMm}{2R} = \frac{GMm}{2R} + \frac{GMm}{R}$$

The first term is the energy required to bring the body from the Earth center to the surface; the second one is the energy required to move it from the surface to infinite. This last one has a nice application.
Let's assume the body gets enough kinetic energy to compensate for the loss of potential energy, that is

$$\frac{GMm}{R} = \frac{1}{2}mv^2$$

Solving with respect to *v* gives

$$v = \sqrt{\frac{2GM}{R}}$$

this is the so called Escape Speed, the minimum speed required for a body to win over the gravitational attraction. Substituting Earth's values we have a speed of around 11000 m/s. A bullet (with no engines) starting from the surface with an initial speed above this value can escape the Earth's gravitational field.

Solving with respect to R and replacing c (the speed of light) for *v* gives the radius of the equivalent-to-Earth Black Hole, a gravitational singularity from which not even light can escape.

$$R_B = \frac{2GM}{c^2}$$

So Earth (replace the values) would become a Black Hole if its mass were to collapse inside a radius of roughly 9 mm.

4) A simple application

Let a massive body be at an height h from the Earth surface and let it fall.
M be the Earth mass, m the body mass, R the Earth radius, G the gravitational constant, g the gravitational acceleration (9.8 m/s^2) on Earth's surface.

Applying the formula for the external potential we have

$$V_h = -\frac{GM}{R+h}$$

$$V_{surface} = -\frac{GM}{R}$$

$$V_h - V_s = -\frac{GM}{R+h} + \frac{GM}{R} = \frac{GMh}{R(R+h)}$$

If h is small compared to R,

$$V_h - V_s \sim \frac{GMh}{R^2}$$

The potential energy difference is

$$\Delta U = (V_h - V_s)m$$

$$\Delta U = \frac{GMmh}{R^2} = F_{12}h$$

since

$$\frac{GM}{R^2} = g$$

$$\Delta U = mgh$$

5) For advanced students

I don't know if this has been tried before but a very similar reasoning can be duplicated for the coulomb force law, with just a different meaning of the constants.
Nothing of relevance changes and you would find very similar functions for the electron (or proton) electric field, assuming they can be modelled as small spheres with a definite radius (or even an average one! Let's not forget the wave-particle duality…).

A potential as shaped in picture three could be used as a cutoff in Quantum Field Theory renormalization, but haven't tested myself this one possibility.

References

[1] Philosophiae Naturalis Principia Mathematica, I.S. Newton, Londini, MDCLXXXVII